



Selected problems of circuit theory

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STANISŁAW OSOWSKI

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Introduction

The textbook introduces concepts of electrical engineering simulation and modeling techniques. The book is a suggested reading for students of the Faculty of Electrical Engineering at Warsaw University of Technology.

1. Signal flow graphs

Signal flow graphs (SFG) are the graphical representation of the linear equations. They fulfill important role in circuit analysis, presenting relations between voltages and currents in a clear visible way. Different forms of graphs are already in use. In circuit analysis the most important is Mason signal flow graph representation and only this form will be considered in this work.

1.1. Basic definitions of SFG

Mason SFG is a graphical way of the flow of signals in the circuit. These signals are represented by the variables appearing in linear description of the system [16,20]. They are treated as the **nodes** in the graph. The nodes are interconnected by the **directed arcs**, which are often called branches. Each arc is described by its **gain** (called also transmittance) representing the proper coefficient of the linear equation.

Example 1.1

Let us consider the system of two linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= F_1 \\ a_{21}x_1 + a_{22}x_2 &= F_2 \end{aligned} \tag{1.1}$$

Mason SFG needs presentation of these equations in an explicit form of succeeding variables. In this case it may look like

$$\begin{aligned} x_1 &= (a_{11} + 1)x_1 + a_{12}x_2 - F_1 \\ x_2 &= a_{21}x_1 + (a_{22} + 1)x_2 - F_2 \end{aligned} \tag{1.2}$$

Now the SFG corresponding to these equations is presented in Fig. 1.1

The internal nodes of the graph are described by the variables x_i ($i = 1, 2$). One node described here by the value 1 represents source node. The only arcs associated with it are going out (the source node cannot have the incoming arcs). The particular nodes are connected with the other nodes by directed arcs, each described by the appropriate gain, representing the coefficient of the linear equation (1.2). Each signal x_i of the graph is equal to the sum of incoming signals weighted by the appropriate gain. Any node of the graph can be treated as an output node (variable).

We can recognize the **loops**, composed of the branches, all of the same direction, forming the closed cycle

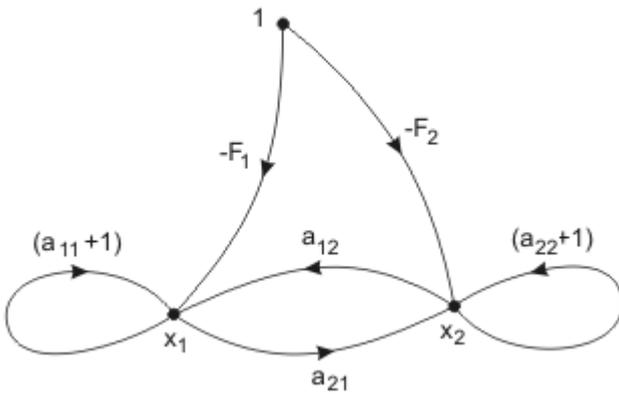


Fig. 1.1. Mason SFG representing equations (1.2).

(without repetition of nodes and branches). The **gain of the loop** is the product of the gains of the branches forming the closed path. In particular the loop may contain only one branch. In such case it is a **self-loop**.

Very important advantage of the Mason SFG is the existence of **Mason gain formula** of the topological nature. It allows calculating any signal x_i of the graph treated as the output node. This formula is defined as the transfer function between the signal of the output node and the source node (in the graph of Fig. 1.1 the source node was assumed as 1). This transfer function may be treated as the gain of the system in relation output/

input. Let us assume notation $T = \frac{X_{wy}}{X_{we}}$, in which $X_{wy}=X_{out}$ represents output signal and $X_{we}=X_{in}$ input signal associated with the source node. According to Mason gain formula the transfer function T is described in general as follows [16,20]

$$T = \frac{\sum_k T_k \Delta_k}{\Delta} \quad (1.3)$$

In this equation Δ is the **main graph determinant** of the graph, which can be presented in the form

$$\Delta = 1 - \sum_i G_i + \sum_{i,j} G_i G_j - \sum_{i,j,k} G_i G_j G_k + \dots \quad (1.4)$$

The coefficients G_i represent the gains of loops in the

graph. The expression for Δ begins with the value of 1. The next term $\sum_i G_i$ represents the sum of the gains of the loops existing in the graph. The following terms $\sum_{i,j} G_i G_j$, etc., represent the gains of non-touching loops combined by two, three, etc. The expansion formula is performed until all non-touching combinations are found. Observe, the following terms of expansion have alternating signs (plus, minus, plus, etc.).

The expression $\sum_k T_k \Delta_k$ in the numerator of equations (1.3) represents all forward paths from the source to the output node. T_k means product of the branch gains from the source to the output. Δ_k is determinant Δ defined for the subgraph which is separated from the k th route. This subgraph is formed by removing the path with all its nodes and arcs. If there are no loops in such subgraph the determinant Δ_k is identically 1. Mason formula will be illustrated on the example of the graph of Fig. 1.1. Applying this formula we get solution for both variables x_1 and x_2 in the following forms

$$T_1 = \frac{x_1}{x_{we}} = \frac{F_1 a_{22} - F_2 a_{12}}{1 - [(a_{11} + 1) + (a_{22} + 1) + a_{12} a_{21}] + (a_{11} + 1)(a_{22} + 1)}$$

$$T_2 = \frac{x_2}{x_{we}} = \frac{-F_1 a_{21} + F_2 a_{11}}{1 - [(a_{11} + 1) + (a_{22} + 1) + a_{12} a_{21}] + (a_{11} + 1)(a_{22} + 1)}$$

In our example the source node is described by $x_{we}=1$, hence the formula delivers direct solution of equations (1.1), because $T_1=x_1$, $T_2=x_2$. Observe that the procedure of solving this system of equations did not involve any mathematical manipulations with equations of the system.

Example 1.2

As the next example consider the SFG presented in Fig. 1.2, of the gains described by the letters a, b, c, \dots, l . The variable x_{we} represents the source node. Let us assume x_5 as an output signal.

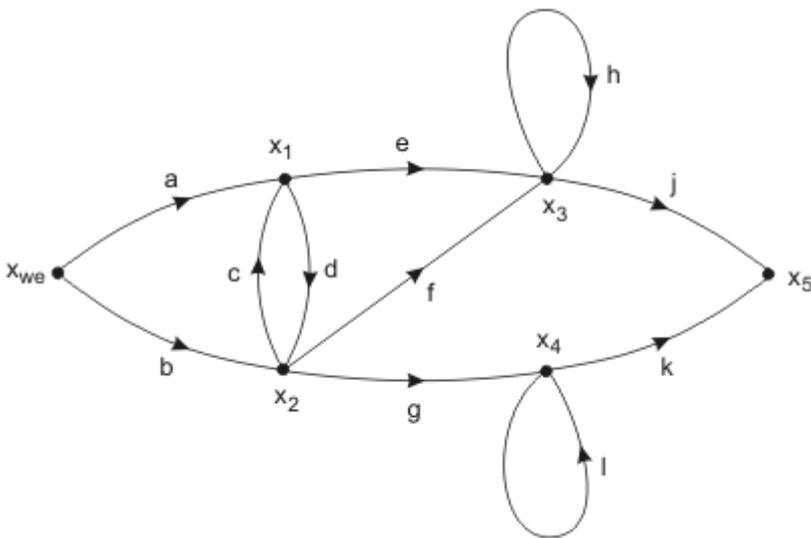


Fig. 1.2. Signal flow graph of the example.

$$T = \frac{x_5}{x_{we}}$$

Transfer function T defined as $\frac{x_5}{x_{we}}$ is now described in the form

$$T = \frac{aej(1-l) + bgk(1-h) + adgk(1-h) + adff(1-l) + bcej(1-l) + bff(1-l)}{1 - (cd + h + l) + (cdh + cdl + hl) - cdhl}$$

The main determinant Δ contains three terms associated with the loops (sum of all loop gains, product of the gains of all combinations of two non-touching loops and product of gains of three non-touching loops). The numerator contains 6 terms in the sum, each corresponding to the particular route from the source to the output node x_5 .

1.2. Application of SFG in analysis of electrical circuits

Mason graph is constructed in general on the basis of the system of linear equations describing the circuit. However, in the case of electrical circuit it is possible to formulate simple rules allowing automatic construction of the graph without describing circuit by the equations. On the other side application of Mason gain formula allows getting solution for any node signal of the circuit. It means that any electrical circuit can be solved

by simply inspecting the structure of the analyzed circuit. It leads to the great simplification in analysis of electrical circuits.

To find proper graph representation of the circuit let us consider some typical connection of circuit elements. Let us start from the connection of onl

y passive elements in the k th node as presented in Fig. 1.3a. The symbols V_1, V_2, \dots, V_n mean node potentials with respect to the reference node (typically the mass).

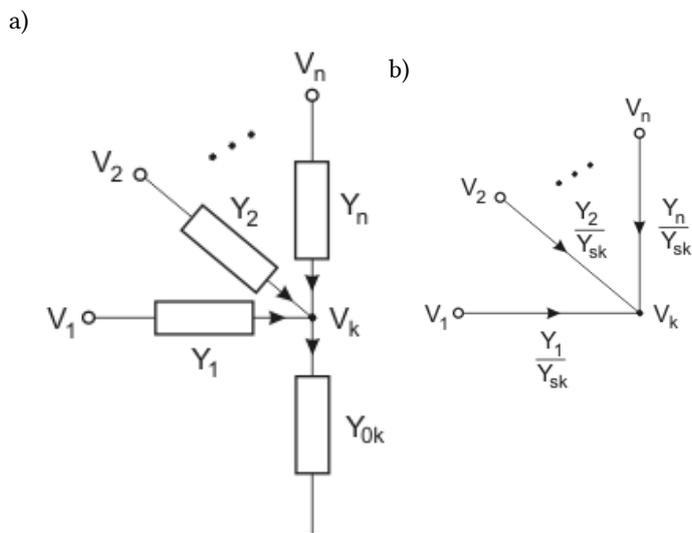


Fig. 1.3. Typical connection of passive elements in the k th node (a) and Mason graph corresponding to such connection (b).

From current Kirchof’s law (nodal description) for this k th node we get

$$Y_1(V_1 - V_k) + Y_2(V_2 - V_k) + \dots + Y_n(V_n - V_k) = Y_{0k}V_k \quad (1.5)$$

After simple mathematical manipulations we get

$$V_k = \frac{Y_1}{Y_{sk}}V_1 + \frac{Y_2}{Y_{sk}}V_2 + \dots + \frac{Y_n}{Y_{sk}}V_n \quad (1.6)$$

where Y_{sk} is the sum of admittances connected to the k th

node, $Y_{sk} = Y_{0k} + \sum_{i=1}^n Y_i$. The equations (1.6) can be associated with the SFG presented in Fig. 1.3b. Observe close similarity of the circuit structure and the graph. Each admittance Y_i ($i = 1, 2, \dots, n$) of the circuit is

represented in the graph by the branch of the gain $\frac{Y_i}{Y_{sk}}$. Each branch going out from any node V_i ($i=1, 2, \dots, n$) is directed toward the node V_k , for which the graph representation is actually constructed. Therefore the graph representation of any node connection in the circuit can be drawn automatically without prior definition of circuit equation for the node.

This simple passive connection of elements can be easily generalized for more complicated connection of elements involving independent current sources and controlled current sources (nodal description accept only current sources, all other types of sources should be converted to current type using Thevenin-Norton equivalent).

Fig. 1.4a shows general form of the node containing not only passive elements but also independent

current source I and current source controlled by the voltage V_{ij} (voltage between node V_i and V_j). From Kirchhoff's current law for this node we get

$$Y_1(V_1 - V_k) + \dots + Y_n(V_n - V_k) + g(V_i - V_j) + I = Y_0 V_k \tag{1.7}$$

After simple mathematical manipulation we get the final description of the voltage in node k .

$$V_k = \frac{Y_1}{Y_{sk}} V_1 + \dots + \frac{Y_n}{Y_{sk}} V_n + \frac{g}{Y_{sk}} V_i - \frac{g}{Y_{sk}} V_j + \frac{1}{Y_{sk}} I \tag{1.8}$$

$$Y_{sk} = Y_{0k} + \sum_{i=1}^n Y_i$$

Once again Y_{sk} is the sum of admittances of passive elements connected to the k th node. The equations (1.8) is represented in the graph form as shown in fig. 1.4b.

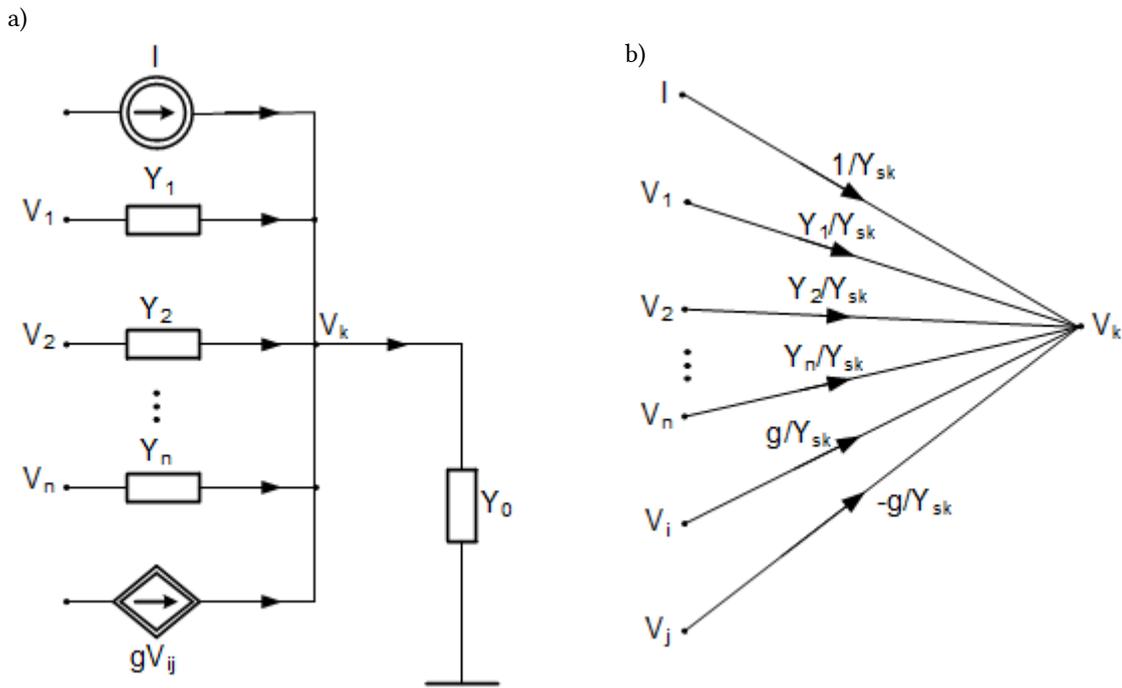


Fig. 1.4. Typical connection of passive elements represented by admittances and current sources in k th node (a) and Mason SFG representing such connection (b).

The interesting simplification in construction procedure is observed in the case of voltage amplifiers, including ideal operational amplifiers (voltage gain tending to infinity). Fig. 1.5 presents the simplified model of voltage amplifier of inverting and non-inverting inputs and the gain A (in the case of ideal operational amplifier the value of A tends to infinity).

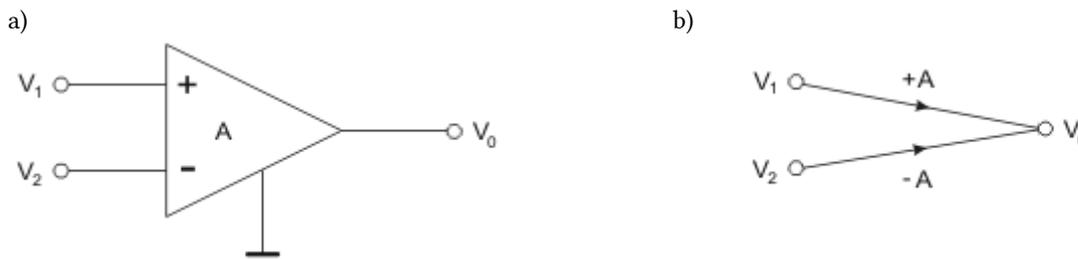


Fig. 1.5. Model of voltage amplifier of inverting and non-inverting inputs (a) and its SFG representation (b)

The output voltage V_o of the amplifier is described by

$$V_o = AV_1 - AV_2 \quad (1.9)$$

The Mason SFG form presented in Fig 1.5b is evident

now.

Very often in practical application we deal with special connection of ideal operational amplifiers (A tending to infinity and infinite input impedance), where the noninverting input is grounded. In such case the SFG of the circuit can be greatly simplified.

Consider such general case presented in Fig. 1.6a. In such connection of op-amp the potential of non-inverting and inverting node is equal zero. Therefore, from Kirchhoff's current law written for inverting node we get

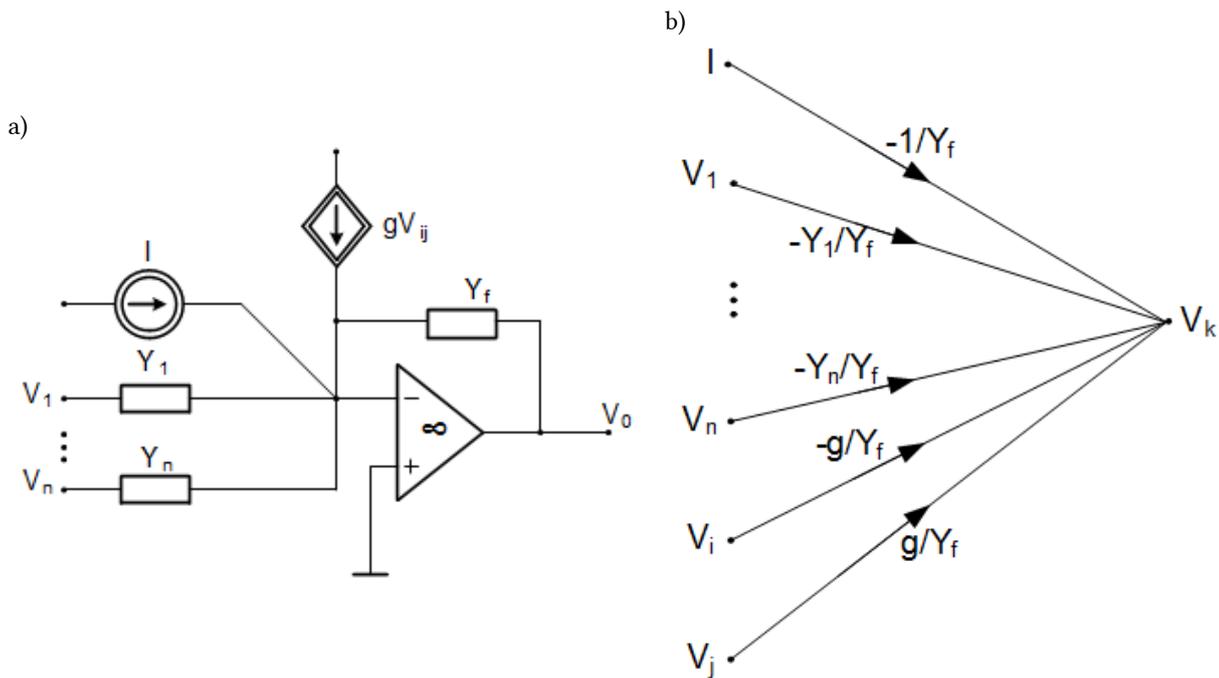
$$Y_1 V_1 + \dots + Y_n V_n + g(V_i - V_j) + I + Y_f V_0 = 0$$

And finally we are able to obtain expression describing directly the output voltage V_0 of the circuit in the following form

$$V_0 = -\frac{Y_1}{Y_f} V_1 + \dots - \frac{Y_n}{Y_f} V_n - \frac{1}{Y_f} I - \frac{g}{Y_f} V_i + \frac{g}{Y_f} V_j$$

The Mason SFG corresponding to such equations is presented in Fig. 1.6b. Once again we can build SFG directly on the basis of inspection of the circuit structure, without writing explicitly Kirchhoff's equations.

(1.11)



Rys. 1.6. Typical general connection of elements containing ideal operational amplifier of infinite gain (a) and Mason SFG corresponding to it (b).

The general procedure of building SFG for the arbitrary circuit structure can be presented in the following way:

1.

Recognize all independent nodes in the circuit and denote their nodal voltages.

2.

Locate all recognized nodes of the future graph in the configuration similar to their positions in the circuit. Recognize the source node, from which only out-going branches are possible (there is no need to build its graph representation).

3.

Step by step build the graph representation for the succeeding nodes using their associated SFG represented in figures 1.3 – 1.6.

4.

In general case for the limited gain A of the amplifier (different from infinity) we must recognize two cases:

4.1

The node under construction is placed on the output of op-amp. In such case use its general model of Fig. 1.5.

4.2

Else, if the node is not on the output of op-amp – use its models presented in Fig. 1.3 or 1.4.

5.

In the case of infinite gain of op-amp you may apply the previous model of op-amp and finally calculate the limit of final expression of transfer function at A tending to infinity, or use the simplified model of op-amp shown in Fig. 1.6.

6.

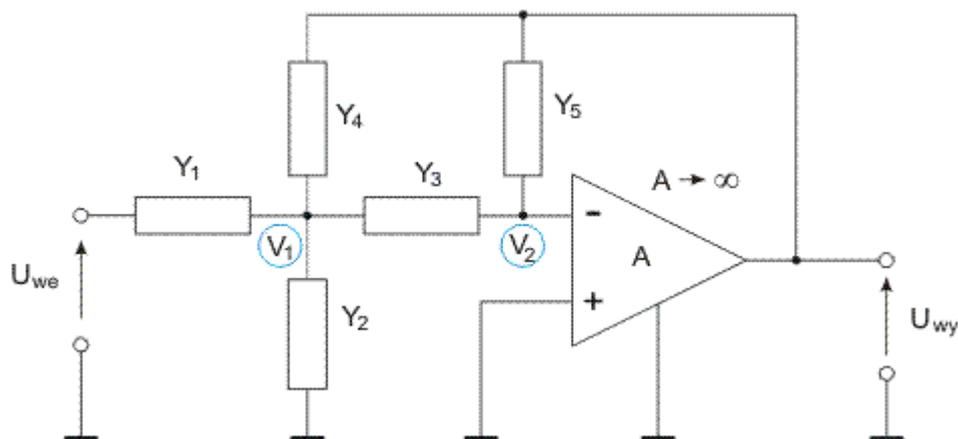
Remember, in constructing SFG representation of particular node the appropriate branches should be directed to this node. After traveling through all nodes of the circuit the final form of the graph is created.

1.3. Examples of SFG application in analysis of circuits with operational amplifiers

1.3.1 Filter structure with multiloop feedback

The procedure described above will be illustrated on the example of multiloop feedback circuit very often used in filter design (Fig. 1.7a). Assume gain of ideal op-amp equal infinity. Because the noninverting node is grounded the node potential V_2 is zero and will not appear in SFG. Therefore, we use the simplified model of op-amp presented in Fig. 1.6b.

a)



b)

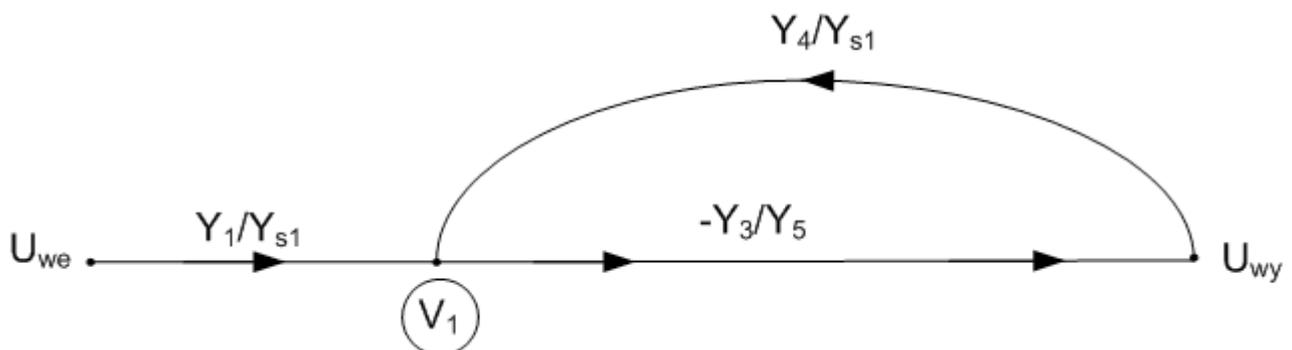


Fig.. 1.7. Multiloop filter structure (a) and Mason SFG corresponding to such circuit (b).

The circuit contains three nodes (V_1 , V_2 and U_{wy}), from which $V_2=0$ and will not appear in the graph. The graph representation of the circuit is created for each node (V_1 , and U_{wy}) in an independent way.

Node V_1 is a typical connection of passive elements and therefore, its graph of Fig. 1.3 is applied. Node $V_2=0$ is excluded from the graph. Graph representation of U_{wy} follows from the model illustrated on Fig. 1.6. Fig. 1.7b shows the final form of the graph. Applying Mason gain formula we get the following transfer function

$$T = \frac{U_{wy}}{U_{we}} = \frac{-\frac{Y_1 Y_3}{Y_{s1} Y_5}}{1 + \frac{Y_3 Y_4}{Y_{s1} Y_5}} = \frac{-Y_1 Y_3}{Y_5 Y_{s1} + Y_3 Y_4} \quad (1.12)$$

where $Y_{s1} = Y_1 + Y_2 + Y_3 + Y_4$. After substituting the expression for Y_{s1} we get the final form of solution

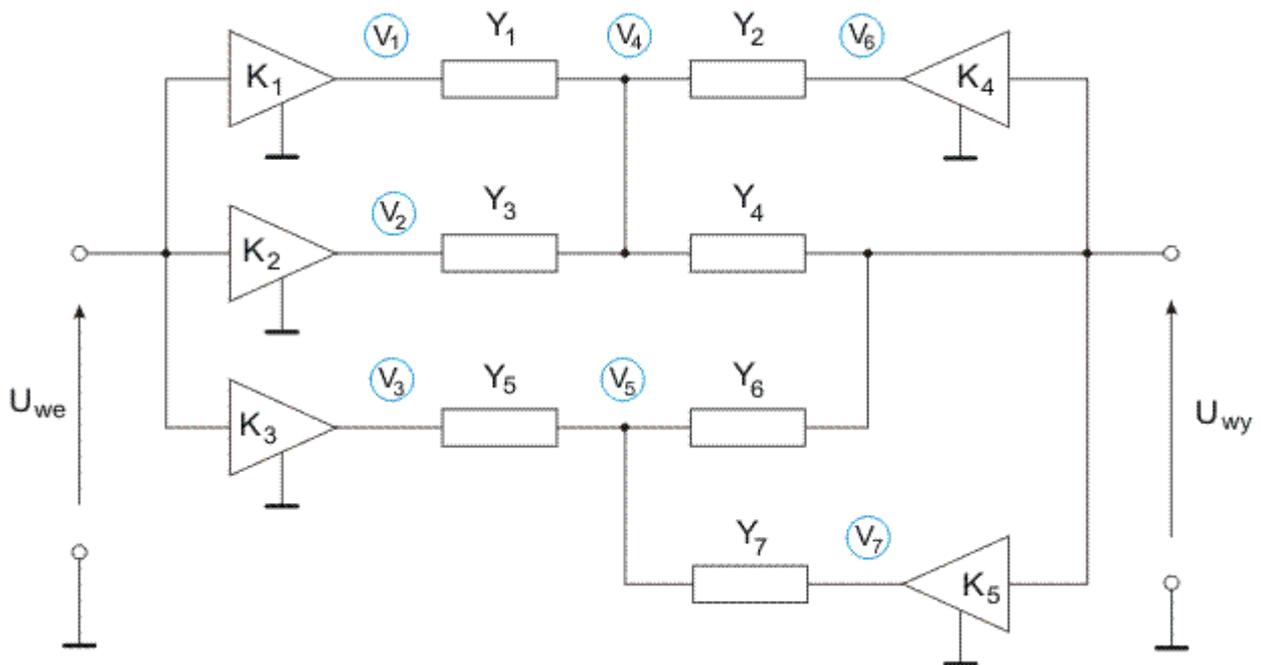
$$T = \frac{-Y_1 Y_3}{Y_5 (Y_1 + Y_2 + Y_3 + Y_4) + Y_3 Y_4} \quad (1.13)$$

Applying proper description of elements in s-domain (for resistor $Y=1/R$, for capacitor $Y=sC$, for inductor $Y=1/sL$) we get the typical transfer function of complex frequency s, $T=T(s)=H(s)$ being the starting point in filter design.

1.3.2 Circuits with voltage amplifiers of finite gain

As an example consider the circuit structure containing five voltage amplifiers of the finite gains: K_1 , K_2 , K_3 , K_4 and K_5 as shown in Fig. 1.8a. Transfer function of the circuit is defined as $T=U_{wy}/U_{we}$ [16].

a)



b)

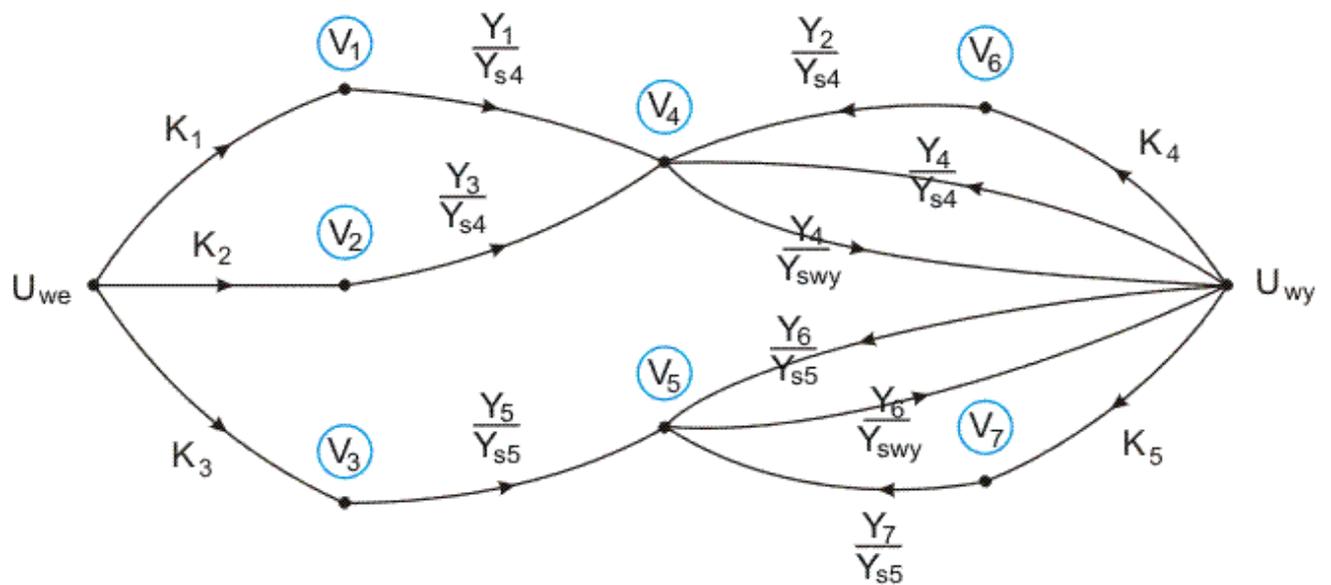


Fig. 1.8. The circuit structure with voltage amplifiers of finite gain (a) and Mason SFG corresponding to this circuit (b).

The SFG, constructed according to the presented procedure, is shown in Fig. 1.8b. After application of Mason gain formula we get transfer function T in the form.

$$T = \frac{K_1 \frac{Y_1}{Y_{s4}} \frac{Y_4}{Y_{swy}} + K_2 \frac{Y_3}{Y_{s4}} \frac{Y_4}{Y_{swy}} + K_3 \frac{Y_5}{Y_{s5}} \frac{Y_6}{Y_{swy}}}{1 - \frac{Y_6^2}{Y_{s5} Y_{swy}} - \frac{K_5 Y_6 Y_7}{Y_{s5} Y_{swy}} - \frac{Y_4^2}{Y_{s4} Y_{swy}} - \frac{K_4 Y_2 Y_4}{Y_{s4} Y_{swy}}} \quad (1.14)$$

where $Y_{s4} = Y_1 + Y_2 + Y_3 + Y_4$, $Y_{s5} = Y_5 + Y_6 + Y_7$, $Y_{s6} = Y_4 + Y_6$. After simplification we get the final form

$$T = \frac{\left(\frac{K_1 Y_1 Y_4 + K_2 Y_3 Y_4}{Y_{s5}} + \frac{K_3 Y_5 Y_6 Y_{s4}}{Y_{s4} Y_{s5} Y_{s6}} - Y_6^2 Y_{s4} - \frac{K_5 Y_6 Y_7 Y_{s4} - Y_4^2 Y_{s5} - K_4 Y_2 Y_4 Y_{s5}}{Y_{s4} Y_{s5}} \right)}{(1.15)}$$

After declaring particular forms of elements (for example resistor $Y=1/R$, capacitor $Y=sC$) we obtain the typical transfer function of complex frequency s , $T=T(s)$.

1.3.3. Biquadratic structure containing three operational amplifiers

Let us consider the so called biquadratic structure built on the basis of three operational amplifiers and RC

$$T(S) = \frac{U_{wy}}{U_{we}}$$

elements, as shown in Fig. 1.9a. The voltage transfer function of the circuit is defined as $T(S) = \frac{U_{wy}}{U_{we}}$. We will build the SFG using model of op-amp of finite gain A , as presented in Fig. 1.5.

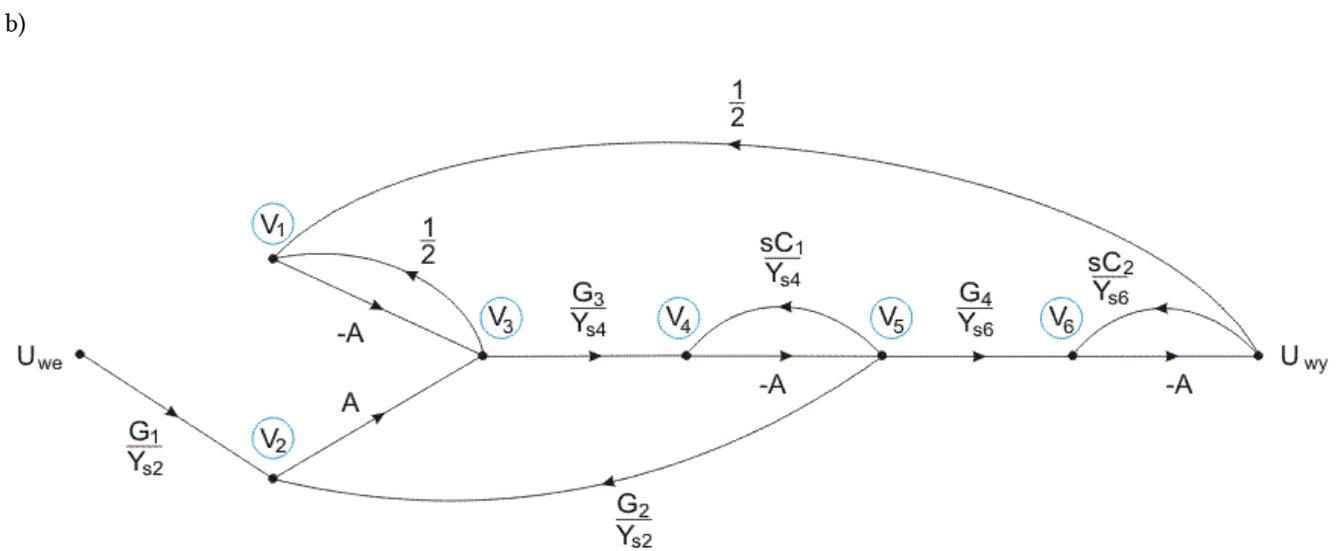
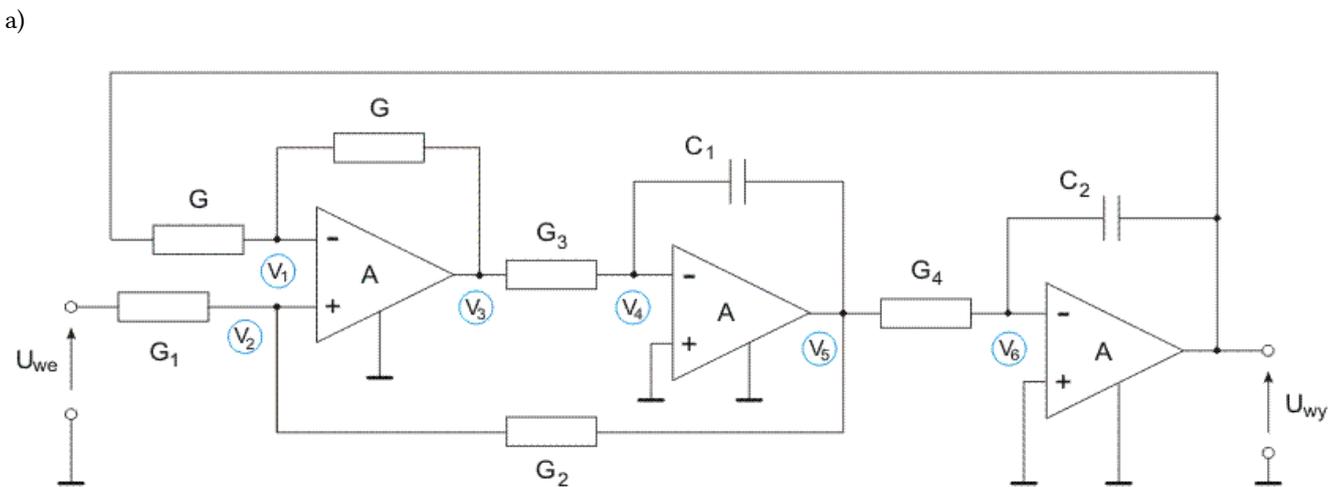


Fig. 1.9. The structure of biquadratic filter (a) and its SF (b)

The Mason SFG built at the assumption of finite gain of operational amplifier is presented in Fig. 1.9b. It contains 5 loops, within which we can recognize some combination of two non-touching loops and even three non-touching loops. Applying Mason gain formula we get

$$T(s) = \frac{A^3 \frac{G_1}{Y_{s2}} \frac{G_3}{Y_{s4}} \frac{G_4}{Y_{s6}}}{M(s)} \quad (1.16)$$

of the denominator $M(s)$ given in the form

$$M(s) = 1 + \left(\frac{1}{2}A + \frac{AsC_1}{Y_{s4}} + \frac{AsC_2}{Y_{s6}} + \frac{A^2G_2G_3}{Y_{s2}Y_{s4}} + \frac{\frac{1}{2}A^3G_3G_4}{Y_{s4}Y_{s6}} \right) + \left(\frac{\frac{1}{2}A^2sC_1}{Y_{s4}} + \frac{\frac{1}{2}A^2sC_2}{Y_{s6}} + \frac{A^2s^2C_1C_2}{Y_{s2}Y_{s4}} + \frac{A^3sG_2G_3C_2}{Y_{s2}Y_{s4}Y_{s6}} \right) + \frac{\frac{1}{2}A^3s^2C_1C_2}{Y_{s2}Y_{s4}Y_{s6}}$$

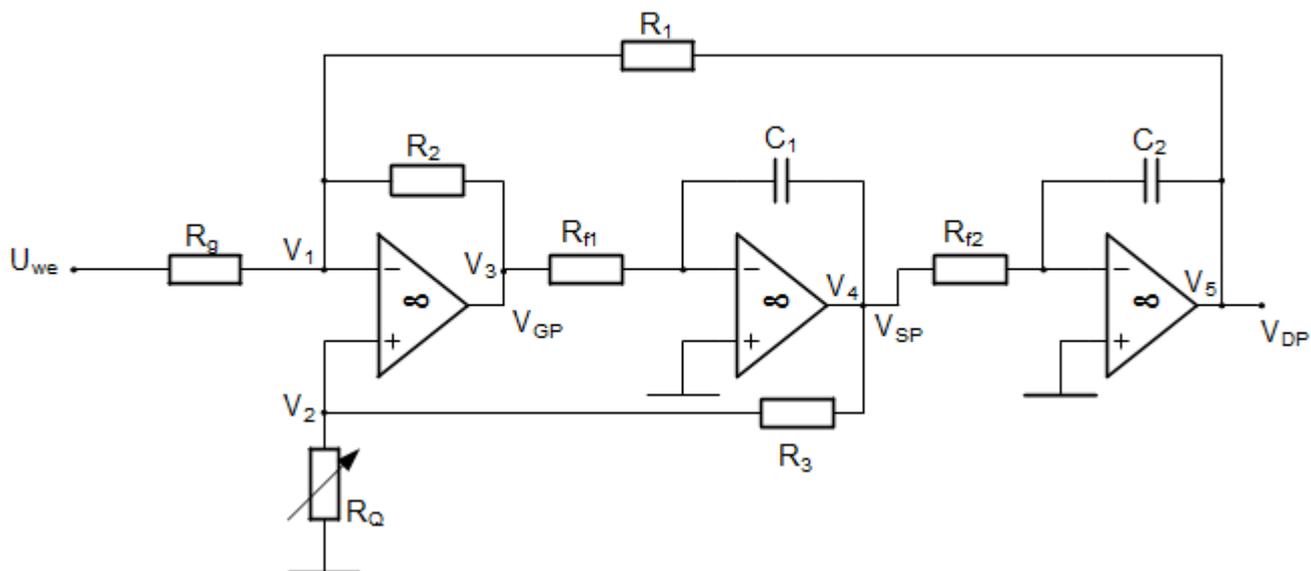
This general solution can be significantly simplified if we assume the gain A tending to infinity, $A \rightarrow \infty$. In such case we get (1.17)

$$T(s) = \frac{G_1G_3G_4}{0,5s^2C_1C_2(G_1 + G_2) + sG_2G_3C_2 + 0,5G_3G_4(G_1 + G_2)}$$

1.3.4. KHN filter structure (1.18)

The next example will consider the biquadratic structure defined by Kervin-Huelsman-Newcomb (KHN) shown in fig. 1.10a containing operational amplifiers of infinite gain [16]. This is the structure similar to the already analyzed in the previous section. This time we apply the simplified model of infinite gain operational amplifier of grounded non-inverting inputs.

a)



b)

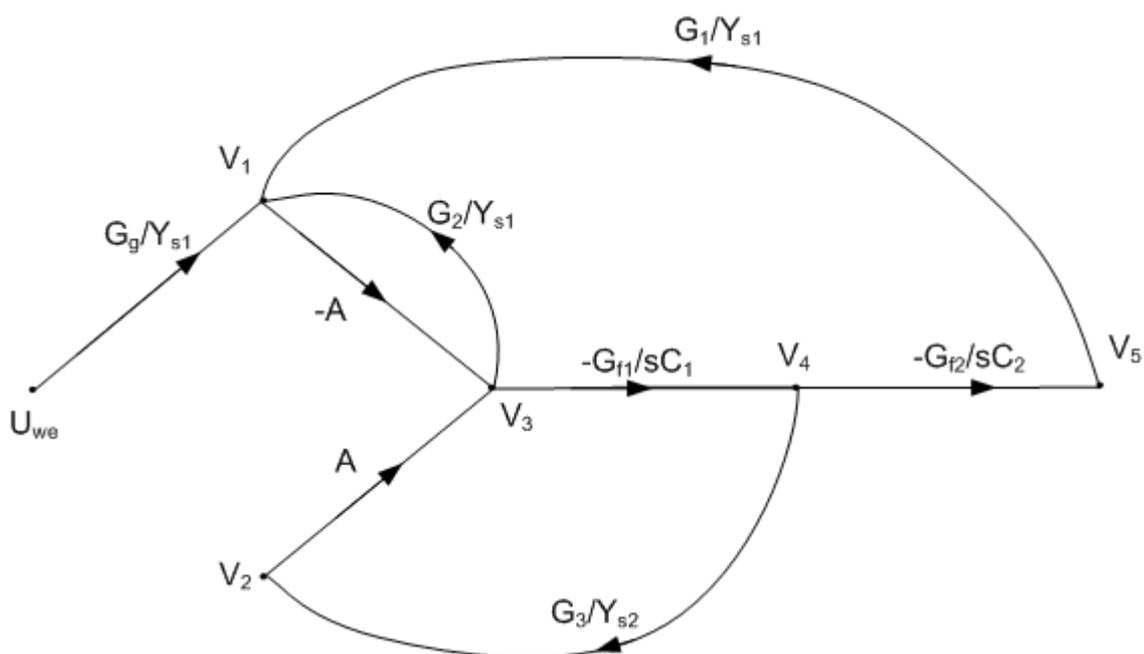


Fig. 1.10. KHN filter structure (a) and Mason SFG corresponding to this structure (b).

Fig. 1.10b presents SFG of the circuit. Applying Mason gain formula we obtain solution in the form of appropriate transfer function. Assuming different output nodes three different transfer functions $T(s)$ can be obtained. They are defined in the following forms

$$T_{LP}(s) = \frac{V_5}{U_{we}} = \frac{-\frac{R_2}{C_1 C_2 R_{f1} R_{f2} R_g}}{M(s)} \quad (1.19)$$

$$T_{BP}(s) = \frac{V_4}{U_{we}} = \frac{s \frac{R_2}{C_1 R_{f1} R_g}}{M(s)} \quad (1.20)$$

$$T_{HP}(s) = \frac{V_3}{U_{we}} = \frac{-s^2 \frac{R_2}{R_g}}{M(s)} \quad (1.21)$$

All transfer functions have the same denominator $M(s)$. It is a polynomial of the second order defined as following (conductance G are inverses of the proper resistances)

$$M(s) = s^2 + s \frac{R_2(G_1 + G_2 + G_g)}{C_1 R_3 R_{f1}(G_3 + G_Q)} + \frac{R_2}{C_1 C_2 R_1 R_{f1} R_{f2}} \quad (1.22)$$

From these expressions it is evident that transfer function $T_{LP}(s)$ represents the lowpass filter, $T_{BP}(s)$ – bandpass filter and $T_{HP}(s)$ – high-pass filter.

1.3.5. FDNR circuit

Mason graphs built for the circuits, according to the presented procedure, are based on nodal description of the circuit and operate only by voltages. In the case when the output variable is a current the additional node of the graph representing the current is needed. The representation of such node is created using the Kirchhoff's law, in which the current is expressed through the nodal voltages.

The example of such case will be presented for the circuit called Frequency Dependent Negative Resistor (FDNR). FDNR represents the input admittance $Y_{we}=Y(s)=Ds^2$, of the positive coefficient D . The structure of FDNR is depicted on Fig. 1.11a. It contains two ideal operational amplifiers, three resistors and two capacitors [4,16].

b)

a)

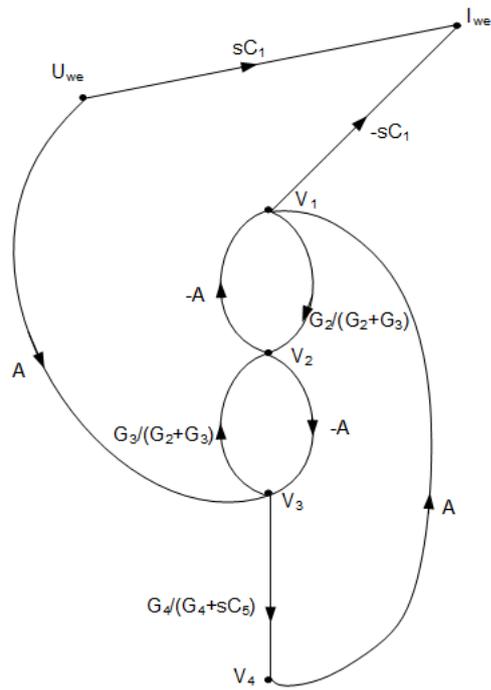
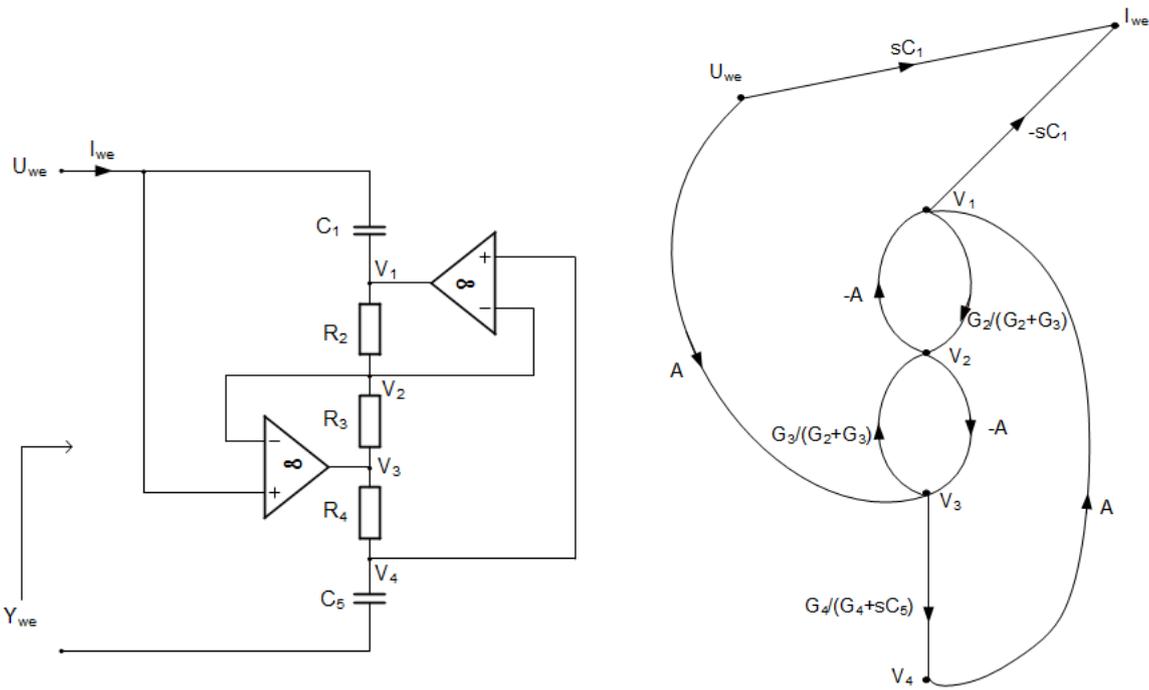


Fig. 1.11. Structure of FDNR (a) and Mason SFG for computing the input admittance (b).

In the case of admittance the output node represents the input current and the source node is the input voltage, since $Y(s)=I_{we}/U_{we}$. Assuming ideal operational amplifiers of infinite input impedance the input current I_{we} can be expressed in the form

$$I_{we}=sC(U_{we}-V_1) \tag{1.23}$$

The SFG for such circuit is built in the same way as for the previous circuits. The only difference is in output node, which is built on the basis of eq. (1.23). Full structure of SFG is shown in Fig. 1.11b, in which the input current is denoted by I_{we} and input voltage by U_{we} . Application of Mason gain formula leads to the solution of the following form, dependent on the arbitrary value of operational amplifier gain A .

$$Y(s) = \frac{sC_1 \Delta + A^2 sC_1 \frac{G_3}{G_2+G_3} - A^2 sC_1 \frac{G_4}{G_4+sC_5}}{\Delta}$$

where the main determinant Δ is given by (1.24)

$$\Delta = 1 + A \frac{G_3}{G_2 + G_3} + A \frac{G_2}{G_2 + G_3} + A^2 \frac{G_2}{G_2 + G_3} \frac{G_4}{G_4 + sC_5}$$

Assuming $A \rightarrow \infty$ the expression for $Y(s)$ is simplified to (1.25)

$$Y(s) = s^2 \frac{C_1 C_5 G_3}{G_2 G_4} = s^2 \frac{C_1 C_5 R_2 R_4}{R_3} \quad (1.26)$$

Comparing this expression with the definition of FDNR $Y(s)=Ds^2$ we find the value of coefficient D as follows

$$D = \frac{C_1 C_5 R_2 R_4}{R_3} \quad (1.27)$$

This value depends only on the parameters of the passive elements (the resistors and capacitors). Observe, that in steady state operation at sinusoidal excitation of the angular frequency ω , the FDNR represents the negative conductance controlled by the frequency, since in such case $Y(s=j\omega)=G=-\omega^2 C_1 C_5 R_2 R_4 / R_3$.

1.4 Exercises

1.

Draw signal flow graph for the system of linear equations

$$x_1 = 2x_1 + 3x_2 + 4x_3$$

$$x_2 = -4x_1 + 5x_3 + 10$$

$$x_3 = x_1 - 2x_2 + 6x_3 + 5$$

Solution:

Graph corresponding to this system of linear equations is presented in Fig. 1.12

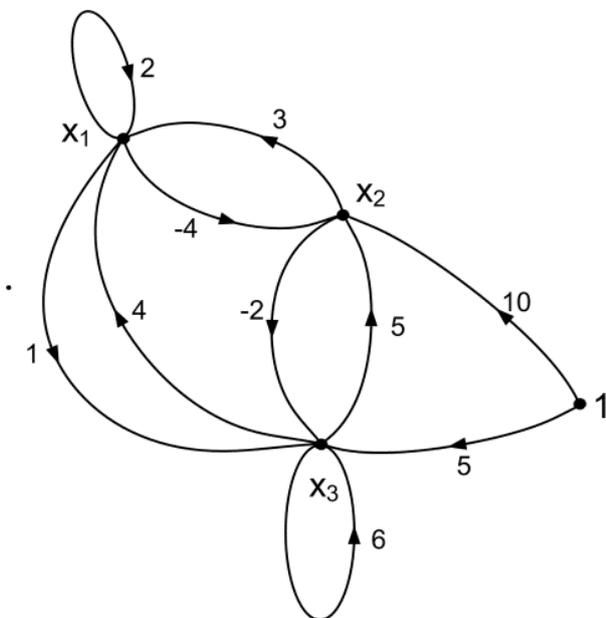


Fig. 1.12 Signal flow graph representing the system of linear equations

1. Draw signal flow graph for the given circuit and calculate V_3

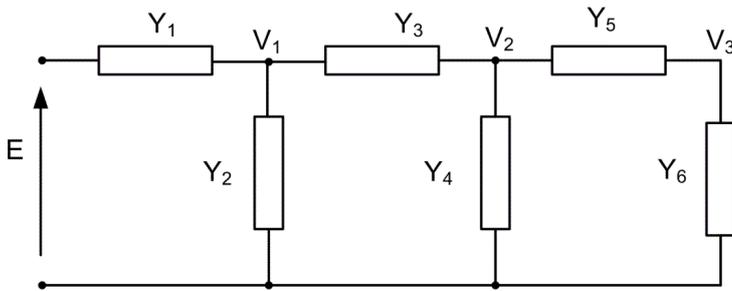


Fig. 1.13a The circuit structure

Solution:

SFG representing the nodal circuit equations for circuit of Fig. 1.2a is shown in Fig. 1.2b.

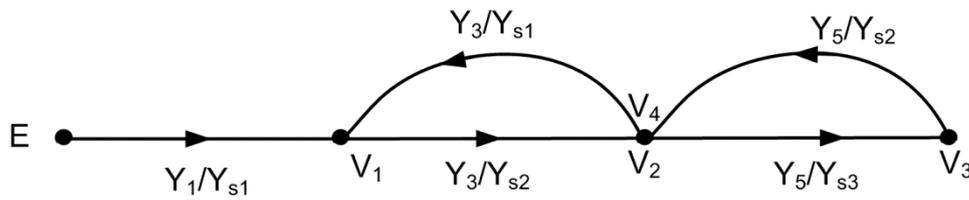


Fig. 1.13b SFG representing the circuit of Fig. 1.13a.

The admittances included in the graph are equal:

$$Y_{s1} = Y_1 + Y_2 + Y_3$$

$$Y_{s2} = Y_3 + Y_4 + Y_5$$

$$Y_{s3} = Y_5 + Y_6$$

From Mason formula we get the expression for voltage V_3 as follows

$$V_3 = E \frac{\sum_k T_k \Delta_k}{\Delta} = E \frac{Y_1 Y_3 Y_5}{1 - Y_3^2 / Y_{s1} Y_{s2} - Y_5^2 / Y_{s2} Y_{s3}} = E \frac{Y_1 Y_3 Y_5}{Y_{s1} Y_{s2} Y_{s3} - Y_3^2 Y_{s3} - Y_5^2 Y_{s1}}$$

Apply signal flow graph to calculate transfer function $T = V_{out} / V_{in}$ at the gain of operational amplifier $A \rightarrow \infty$

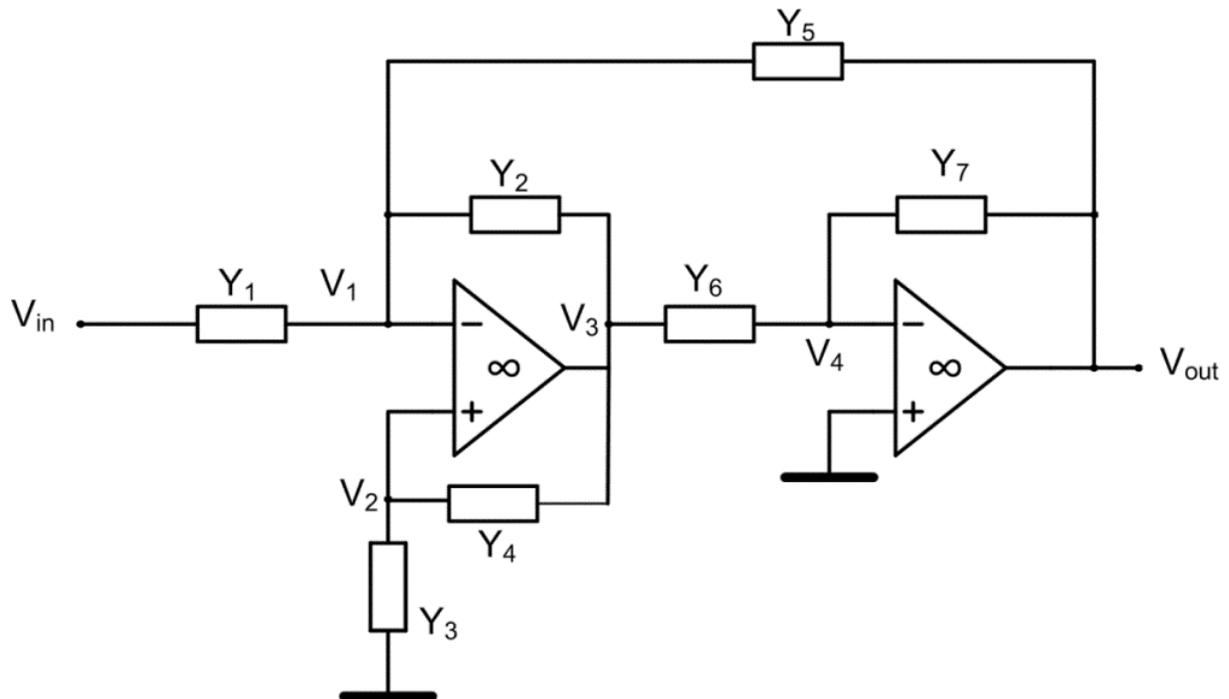


Fig. 1.14a The circuit structure subject to analysis

Solution:

SFG corresponding to the circuit structure of Fig. 1.3a is presented in Fig. 1.3b, where $Y_{s1}=Y_1+Y_2+Y_5$.

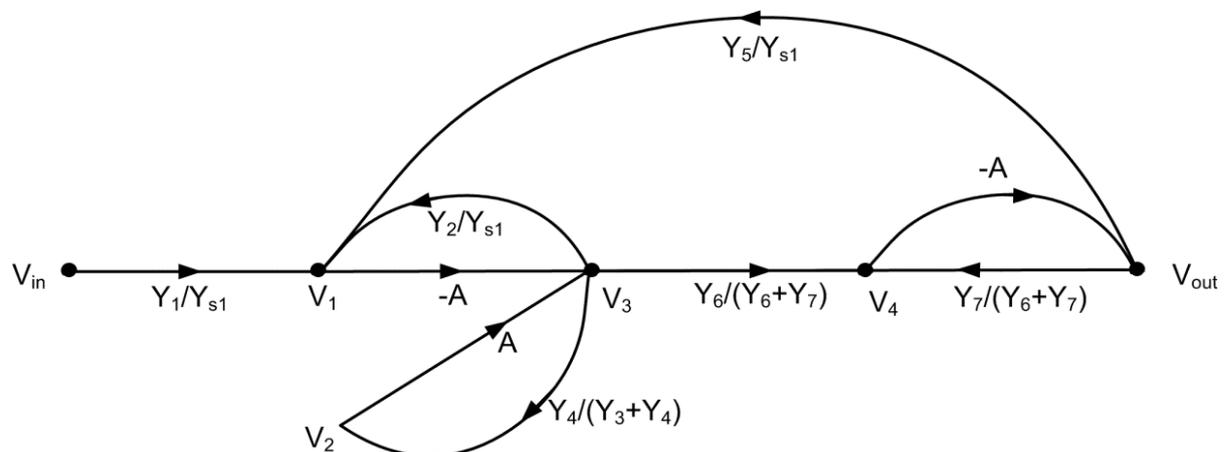


Fig. 1.14b SFG corresponding to the circuit of Fig. 1.14a.

From Mason formula we obtain

$$T = \frac{\sum_k T_k \Delta_k}{\Delta} = \frac{A^2 \frac{Y_1}{Y_{s1}} \frac{Y_6}{Y_6+Y_7}}{1 - \left(-\frac{AY_2}{Y_{s1}} + \frac{AY_4}{Y_3+Y_4} - \frac{AY_7}{Y_6+Y_7} + \frac{A^2 Y_5 Y_6}{Y_{s1}(Y_6+Y_7)} \right) + \left(\frac{A^2 Y_2 Y_7}{Y_{s1}(Y_6+Y_7)} - \frac{A^2 Y_4 Y_7}{(Y_3+Y_4)(Y_6+Y_7)} \right)}$$

Assuming $A \rightarrow \infty$ we get

$$T_\infty = \frac{Y_1 Y_6 (Y_3 + Y_4)}{Y_2 Y_7 (Y_3 + Y_4) - Y_5 Y_6 (Y_3 + Y_4) - Y_4 Y_7 (Y_1 + Y_2 + Y_5)}$$

1.5 Basic definitions

Branch – the directed arch joining two nodes in the graph.

Branch gain – the gain parameter associated with the branch. The output node signal of the branch is its input node signal multiplied by this gain.

FDNR – frequency dependent negative resistor, i.e., the electronic circuit implementing negative resistance controlled by the frequency.

Forward path – a path from an input node to an output node in which no node is touched more than once.

Gain of graph – the ratio of output signal to the input (source) signal calculated by Mason gain formula. It is also called transfer function.

Graph determinant – the mathematical expression used in Mason gain formula. It is formed on the basis of the loop gains existing in the graph.

Graph model of circuit – graph representation of the circuit equations, built usually by inspection of the circuit without declaring the Kirchhoff's description in an explicit way.

Loop – a closed path of directed branches. It originates and ends on the same node, and no node is touched more than once).

Loop gain – the product of the gains of all the branches in the loop.

Mason's gain formula – the method for finding the [transfer function](#) from the source to the output node of a linear [signal-flow graph](#).

Non-touching loops – the loops having no common nodes.

Path – the continuous set of branches traversed in the direction indicated by the branch arrows.

Path gain – the product of the gains of all the branches in the path.

Self-loop – the branch, which begins and ends at the same node without entering other nodes.

Signal flow graph – the graphical representation of the linear system of equations in which variables are represented by nodes and coefficients by the gains of archs joining two nodes.

2. Synthesis of passive networks

The task of synthesis is to design the circuit structure and element values in such a way that the circuit possesses the required transfer function in s-domain [1,13,16]. The starting point is the given form of transfer function, which should be implemented using the required types of circuit elements. In this chapter we will consider the realization of the impedance or admittance transfer function implemented using the set of passive elements (resistors, inductors and capacitors). We assume ideal passive elements described by their impedances: resistor by resistance R , capacitor by impedance $Z_C=1/sC$, inductor by impedance $Z_L=sL$ (including the magnetic coupling). In special cases gyrator (special passive 2-port network) is also allowed in practical implementation.

3.1. Definition of passive elements

Circuit elements can be classified into different categories. Among them we can recognize [4,7,16]:

- One-port elements – they represent the simplest components, that have only two terminals to connect to. Examples: resistors, capacitors, inductors, diodes.
- Multiport elements – the elements that have more than two terminals. Example: a transformer with four separate windings has 8 terminals and could be represented as a four-port element.
- Two-port elements – the multiport elements of two ports: one is regarded as input and the second as output. Examples: gyrator, transistors.

In this chapter we will be mainly involved in the synthesis task using one-port (or 2-terminal) passive elements. These are elements which are not a source of energy. The examples: resistances, capacitances and inductances. Formally the 2-terminal element is called passive, if the energy $w(t)$ of the element satisfies the condition [1,13,16]

$$w(t) = \int_{t_0}^t u(\tau)i(\tau)d\tau \geq 0 \quad (3.1)$$

at initial condition $w(t_0)=0$. On the contrary the element is called active, if at any point t the following condition is satisfied

$$w(t) = \int_{t_0}^t u(\tau)i(\tau)d\tau < 0 \quad (3.2)$$

The passivity definition may be extended also to N-ports. In such case the condition (3.1) is taken over all ports and written in the form

$$w(t) = \int_{t_0}^t \sum_{k=1}^N u_k(\tau) i_k(\tau) d\tau \geq 0 \quad (3.3)$$

The important property of passive elements is their stability. Passive elements are always stable, which means that their currents and voltages are always limited.

3.2. Realizability conditions for passive RLC circuits

Let us assume the circuit is described by the impedance $Z(s)$ or admittance $Y(s)$ of the rational form. Observe that $Y(s)=1/Z(s)$. It means that poles of $Z(s)$ are automatically zeros of $Y(s)$ and vice versa. Assume the rational form of $Z(s)$ as follows

$$Z(s) = \frac{L(s)}{M(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \quad (3.4)$$

of the real and nonnegative coefficients a_i, b_i of the denominator and numerator, respectively.

3.2.1. RLC circuits

The impedance $Z(s)$ given by (3.4) is realizable in the class of RLC elements if the following conditions are met [1,16]

1. All coefficients a_i, b_i of $Z(s)$ are real and positive.
2. All zeros (the roots of numerator) and poles (the roots of denominator) are located in left half-plane of complex variable s . In the case of location on the imaginary axis they are single. The residues in all poles are positive.
3. The degree of $L(s)$ and $M(s)$ might be different by at most one.
4. For $\text{Re}(s) \geq 0$ the condition $\text{Re}[Z(s)] \geq 0$ should be satisfied.

Example 3.1

We will check the realizability condition of RLC circuit described by $Z(s)$ given as follows

$$Z(s) = \frac{L(s)}{M(s)} = \frac{s^2 + 6s + 8}{s^2 + 4s + 3}$$

- The first condition is automatically satisfied, since all coefficients are positive.
- The zeros: $z_1=-2, z_2=-4$ and poles $p_1=-1$ and $p_2=-3$ are located in left half-plane, hence, the second condition is also satisfied.
- The degrees of $L(s)$ and $M(s)$ are the same, so the third condition is also met.
- To check the last condition we should determine $Z(s)$ for $s=\sigma+j\omega$. After substitution this value of s to $Z(s)$ we get

$$Z(\sigma + j\omega) = \frac{(\sigma + j\omega)^2 + 6(\sigma + j\omega) + 8}{(\sigma + j\omega)^2 + 4(\sigma + j\omega) + 3} = \frac{(\sigma^2 + 8 - \omega^2 + 6\sigma) + j(2\sigma\omega + 6\omega)}{(\sigma^2 + 3 - \omega^2 + 4\sigma) + j(2\sigma\omega + 4\omega)}$$

It means that

$$\operatorname{Re} [Z(\sigma + j\omega)] = \frac{\sigma^4 + 10\sigma^3 + \sigma^2(35 + 2\omega^2) + \sigma(50 + 10\omega^2) + \omega^4 + 13\omega^2 + 24}{(\sigma^2 + 3 - \omega^2 + 4\sigma)^2 + (2\sigma\omega + 4\omega)^2} > 0$$

At $\sigma \geq 0$ we obtain $\operatorname{Re} [Z(\sigma + j\omega)]$ which is always positive. Hence also the fourth condition is satisfied. Therefore, the considered impedance form $Z(s)$ is realizable using RLC elements.

Sometimes the other realizability conditions are used in practice. They may be presented in the following form.

1. All coefficients a_i, b_i of $Z(s)$ are real and positive.
2. The degree of $L(s)$ and $M(s)$ might be different by at most one.
3. The rational function constructed on the basis of even parts M_p, L_p of polynomials $M(s)$ and $L(s)$ and odd parts M_n, L_n of the original polynomials

$$F(s) = \frac{M_p + L_p}{M_n + L_n} \quad (3.5)$$

presented in the form of continued fractions contains only elements of real values or linear form of s

1. The value of the function $M_p L_p - M_n L_n \geq 0$ is non-negative for any $s = j\omega$.

Example 3.2

Let us consider impedance $Z(s)$ given as follows

$$Z(s) = \frac{L(s)}{M(s)} = \frac{s^2 + 8s + 12}{s^3 + 9s^2 + 8s}$$

- The first two conditions are evidently satisfied.
- To check the third one observe that

$$M_p = 9s^2$$

$$L_p = s^2 + 12$$

$$M_n = s^3 + 8s$$

$$L_n = 8s$$

Hence the function (3.5) takes the form of continued fraction

$$F(s) = \frac{M_p + L_p}{M_n + L_n} = \frac{10s^2 + 12}{s^3 + 16s} = \frac{1}{0.1s + \frac{1}{\frac{25}{37}s + \frac{1}{\frac{74}{60}s}}}$$

All variable coefficients in this expansion are positive. Hence, the third condition is also satisfied.

- To check the fourth condition we must calculate the value of $M_p L_p - M_n L_n$ for $s = j\omega$. This function is described as follows

$$M_p L_p - M_n L_n = 9s^2(s^2 + 12) - 8s(s^3 + 8s) = s^4 + 44s^2 \Big|_{s=j\omega} = \omega^4 - 44\omega^2$$

It is evident now, that for same values of ω ($\omega < 44$) the value of this function is negative. Therefore, the considered $Z(s)$ is not realizable in the class of RLC elements.

3.2.2. Realizability in class of RL elements

The realizability conditions are significantly simplified if only 2-element classes are considered. In the case of RL elements the following conditions should be satisfied.

1. Zeros and poles are single, located on the negative real axis and alternate
2. The closest to the origin is the zero of $Z(s)$ or pole of $Y(s)=1/Z(s)$.

Example 3.3

Consider the impedance function $Z(s)$ given as follows

$$Z(s) = \frac{s^3 + 7s^2 + 10s}{s^2 + 4s + 3}$$

The zeros of this function are $z_1=0$, $z_2=-2$ and $z_3=-5$, the poles: $p_1=-1$, $p_2=-3$. It is evident that all conditions are met. The closest to the origin is zero, hence, the function $Z(s)$ is realizable in the class of RL elements.

Example 3.4

As the next example consider the function $Z(s)$ given as follows

$$Z(s) = \frac{s^2 + 3s + 2}{s^3 + 7s^2 + 12s}$$

The zeros of this function are: $z_1=-1$, $z_2=-2$, and poles $p_1=0$, $p_2=-3$, $p_3=-4$. This time the closest to the origin is the pole, moreover, the poles and zeros do not alternate. Hence this function is not realizable in the class of RL elements.

3.2.3. Realizability in class of RC elements

The conditions of realizability of $Z(s)$ in the class of RC elements can be expressed in the following form

1. Zeros and poles are single, located on the negative real axis and alternate.
2. The closest to the origin is the pole of $Z(s)$ or zero of $Y(s)=1/Z(s)$.

The only difference to RL elements is that this time the pole is closest to the origin (in the previous case it was zero).

Example 3.5

Let us consider $Z(s)$ given as follows

$$Z(s) = \frac{s^2 + 4s + 3}{s^3 + 8s^2 + 12s}$$

The zeros of the function are: $z_1=-1$, $z_2=-3$, and poles: $p_1=0$, $p_2=-2$ oraz $p_3=-6$. It is evident that all conditions are met. The closest to the origin is pole, hence, the function $Z(s)$ is realizable in the class of RC elements.

Example 3.6

As the next example consider the function of the form

$$Z(s) = \frac{(3/2s^2 + 3/4) s}{s^4 + 8s^2 + 12s}$$

The zeros of $Z(s)$ are: $z_1=0$, $z_2=-j0.707$, $z_3=j0.707$ and poles: $p_1=j$, $p_2=-j$, $p_3=-j0.5$, $p_4=j0.5$. The zeros and poles are complex, which is not allowed in RL or RC classes. Therefore, this function is not realizable either in RL or in RC class.

3.2.4. Realizability in class of LC elements

The function $Z(s)$ is realizable in the class of LC elements if the following conditions are met.

1. The immittance (impedance or admittance) is expressed as the ratio of two polynomials, from which one contains only even and the second only odd powers.
2. The degrees of $L(s)$ and $M(s)$ should be different by one and both polynomials should contain all succeeding powers different from zero.
3. Zeros and poles are single, placed on imaginary axis $j\omega$, and alternate.

Example 3.7

Let us consider the impedance function

$$Z(s) = \frac{s^3 + 0.5s}{s^4 + 1.25s^2 + 0.25}$$

The zeros of this function are: $z_1=0$, $z_2=-j0.707$, $z_3=j0.707$ and poles: $p_1=j$, $p_2=-j$, $p_3=-j0.5$, $p_4=j0.5$. All conditions of realizability in LC class are met. The function may be realized as connection of LC elements.

3.3. Canonical structures of circuits in 2-class elements

One of the synthesis methods is to define the basic 2-element structures, define their basic immittances, expand the given function into single fractions corresponding to these basic structures and associate them with the final circuit in the form of series connection for impedance $Z(s)$ or parallel connection for admittance $Y(s)$.

In the case of RL elements we have two types of impedances: $Z_R = R$ for resistors and $Z_L = sL$ for inductors. Fig. 3.1 presents two connections of such elements: parallel RL for impedance $Z(s)$ and series for admittance $Y(s)$.

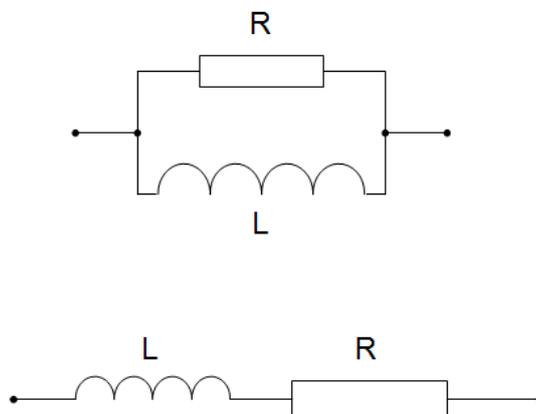


Fig. 3.1 Canonical structures of RL elements.

Parallel structure of Fig. 3.1 is described by impedance

$$Z_{RL} = \frac{sR}{s + R/L} \tag{3.6}$$

and series structure of Fig. 3.1 by the admittance function

$$Y_{RL} = \frac{1/L}{s + R/L} \tag{3.7}$$

In the case of RC elements the canonical structures are presented in Fig. 3.2. The parallel connection is used in realization of impedance $Z(s)$ and series connection for admittance $Y(s)$.

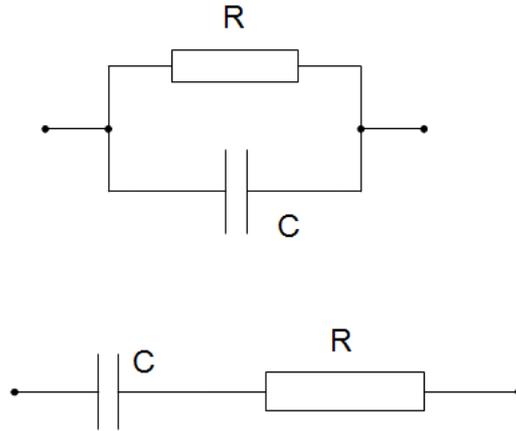


Fig. 3.2 Canonical structures for RC circuits RC.

Parallel structure of Fig. 3.2a is described by the impedance $Z_{RC}(s)$

$$Z_{RC} = \frac{1/C}{s + 1/RC} \tag{3.8}$$

and series connection of elements (Fig. 3.2) by the admittance $Y_{RC}(s)$

$$Y_{RC} = \frac{s/R}{s + 1/RC} \tag{3.9}$$

In an identical way we describe the canonical structures of LC elements. In this case we deal with the impedances of elements $Z_L=sL$ for inductor and $Z_C=1/sC$ for capacitor. The canonical structures of LC elements are presented in Fig. 3.3.

The parallel structure of Fig. 3.3 is described by impedance

$$Z_{LC} = \frac{s/C}{s^2 + 1/LC} \tag{3.10}$$

and the series structure of Fig. 3.3 by admittance

$$Y_{LC} = \frac{s/L}{s^2 + 1/LC} \tag{3.11}$$

Presenting the given $Z(s)$ or $Y(s)$ by the sum of fractions corresponding to any canonical form allows realizing the function as the series connection of canonical structures in the case of impedance function or parallel connection in the case of admittance function.

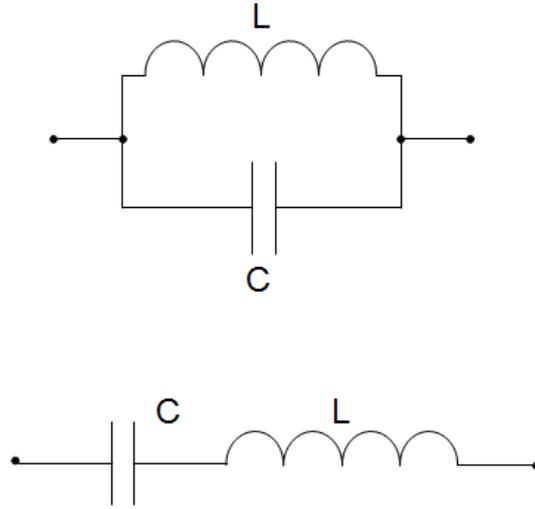


Fig. 3.3 Canonical structures of LC elements.

3.4. Foster method

Foster method is applicable to two-element circuits only. The main idea behind it is to decompose the given function into the sum of **partial fractions** corresponding to the appropriate canonical 2-element structures and combine these canonical structures in the series (for impedance) or parallel (for admittance) connection [1,16].

• 4.1. Foster method in class of RL elements

The impedance of RL class can be decomposed into the sum of the following terms

$$Z(s) = a_0 + d_0 s + \sum_k \frac{d_k s}{s - p_k} \quad (3.12)$$

where p_k denotes the poles of the function under synthesis. This expression corresponds to the canonical RL sections of Fig. 3.4.

$$Z_{RL}(s) = R_0 + sL_0 + \sum_k \frac{R_k s}{s + \frac{R_k}{L_k}} \quad (3.13)$$

Each term of eq. (3.12) can be calculated using the following expressions

$$\begin{aligned} a_0 &= \lim_{s \rightarrow 0} Z(s) \\ d_0 &= \lim_{s \rightarrow \infty} \frac{Z(s)}{s} \\ d_k &= \lim_{s \rightarrow p_k} \frac{Z(s)(s - p_k)}{s} \end{aligned} \quad (3.14)$$

Comparing expansions (3.12) and (3.13) allows identifying the values of RL elements. Namely,

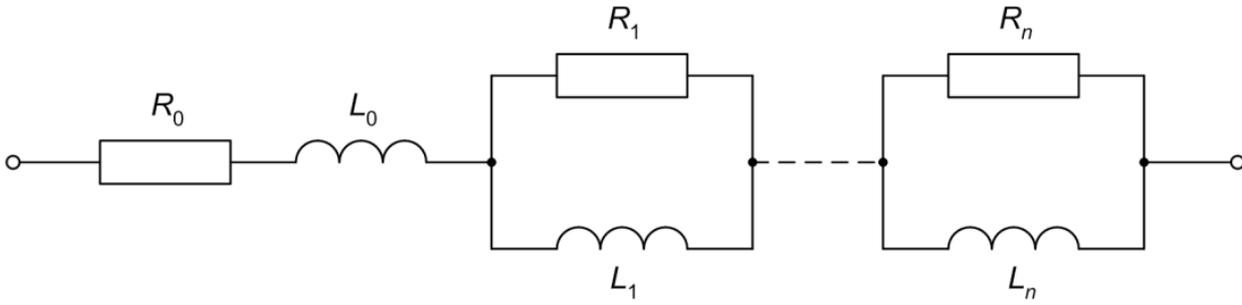


Fig. 3.4 The general RL structure of Foster method in realization of impedance $Z(s)$.

$$\begin{aligned}
 R_0 &= a_0 \\
 L_0 &= d_0 \\
 R_k &= d_k \\
 L_k &= -d_k/p_k
 \end{aligned}
 \tag{3.15}$$

Example 3.8

Consider the given function $Z(s)$ realizable in RL class of elements

$$Z(s) = \frac{s^3 + 7s^2 + 10s}{(s + 1)(s + 3)}$$

The function possesses two poles: $p_1=-1$ and $p_2=-3$. According to the presented above procedure we get

$$\begin{aligned}
 a_0 &= \lim_{s \rightarrow 0} Z(s) = 0 \\
 d_0 &= \lim_{s \rightarrow \infty} \frac{Z(s)}{s} = 1 \\
 d_1 &= \lim_{s \rightarrow -1} \frac{Z(s)(s+1)}{s} = 2 \\
 d_2 &= \lim_{s \rightarrow -3} \frac{Z(s)(s+3)}{s} = 1
 \end{aligned}$$

It mean, that the impedance $Z(s)$ can be presented in the form

$$Z(s) = s + \frac{s}{s + 3} + \frac{2s}{s + 1}$$

From (3.14) we get:

$$\begin{aligned}
 R_0 &= a_0 = 0 \\
 L_0 &= d_0 = 1 \\
 R_1 &= d_1 = 2 \\
 L_1 &= -d_1/p_1 = 2 \\
 R_2 &= d_2 = 1 \\
 L_2 &= -d_2/p_2 = 1/3
 \end{aligned}$$

Full circuit structure realizing given $Z(s)$ is shown in Fig. 3.5.

In the case of admittance function $Y(s)$ we decompose it into the form corresponding to series connection of RL elements (Fig. 3.1b). It means that the final form of $Y(s)$ will be presented as follows

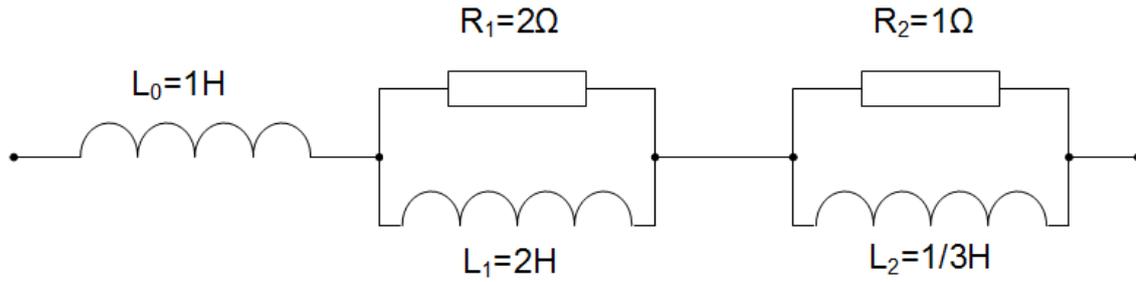


Fig. 3.5 The general RL structure, representing the given function $Z(s)$.

$$Y_{RL}(s) = G_0 + \frac{1}{sL_0} + \sum_k \frac{1/L_k}{s + \frac{R_k}{L_k}} \quad (3.16)$$

Hence, the expansion of $Y(s)$ should be defined in the form

$$Y(s) = a_0 + \frac{d_0}{s} + \sum_k \frac{d_k}{s - p_k} \quad (3.17)$$

The coefficients of this expansion are defined as

$$\begin{aligned} a_0 &= \lim_{s \rightarrow \infty} Y(s) \rightarrow G_0 = a_0 \\ d_0 &= \lim_{s \rightarrow 0} sY(s) \rightarrow L_0 = 1/d_0 \\ d_k &= \lim_{s \rightarrow p_k} Y(s)(s - p_k) \rightarrow L_k = 1/d_k, R_k = -d_k/p_k \end{aligned} \quad (3.18)$$

The last step is to associate the elements of (3.16) with the form of (3.17) and connect the canonical RC structures in the parallel fashion.

• 4.2. Foster method in class of RC elements

The impedance $Z(s)$ should be now decomposed into the form

$$Z(s) = a_0 + d_0/s + \sum_k \frac{d_k}{s - p_k} \quad (3.19)$$

where p_k denotes the poles of the function under synthesis (negative values). This expression corresponds to the canonical RC sections of Fig. 3.6 and is described by

$$Z_{RC}(s) = R_0 + \frac{1}{sC_0} + \sum_k \frac{1/C_k}{s + \frac{1}{R_k C_k}} \quad (3.20)$$

The coefficients in expansion (3.19) can be calculated using the formulas

$$\begin{aligned} a_0 &= \lim_{s \rightarrow \infty} Z(s) \\ d_0 &= \lim_{s \rightarrow 0} sZ(s) \\ d_k &= \lim_{s \rightarrow p_k} Z(s)(s - p_k) \end{aligned} \quad (3.21)$$

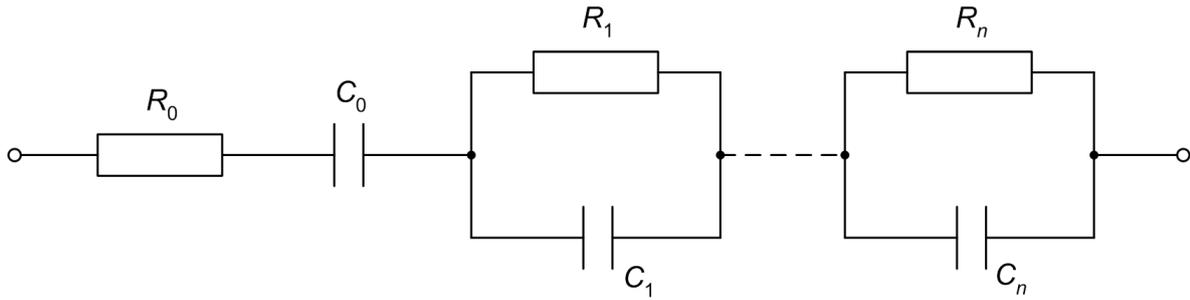


Fig. 3.6 The general structure of RC elements in Foster synthesis method.

Comparing (3.19) with (3.20) allows identifying the values of RC elements

$$\begin{aligned}
 R_0 &= a_0 \\
 C_0 &= 1/d_0 \\
 R_k &= -d_k/p_k \\
 C_k &= 1/d_k
 \end{aligned}
 \tag{3.22}$$

Example 3.9

Let us illustrate the method by $Z(s)$ given in the form

$$Z(s) = \frac{s^2 + 4s + 3}{s(s + 2)(s + 6)}$$

There are three poles: $p_1=0$, $p_2=-2$ and $p_3=-6$. According to Foster procedure we get

$$\begin{aligned}
 a_0 &= \lim_{s \rightarrow \infty} Z(s) = 0 \\
 d_0 &= \lim_{s \rightarrow 0} sZ(s) = 1/4 \\
 d_1 &= \lim_{s \rightarrow -2} Z(s)(s + 2) = 1/8 \\
 d_2 &= \lim_{s \rightarrow -6} Z(s)(s + 6) = 5/8
 \end{aligned}$$

The given $Z(s)$ can be presented as the sum of the first order partial fractions of rational form

$$Z(s) = \frac{1}{4s} + \frac{1/8}{s + 2} + \frac{5/8}{s + 6}$$

Hence, the elements of RC structure of Fig. 3.6 are calculated as follows:

$$\begin{aligned}
 R_0 &= a_0 = 0 \\
 C_0 &= 1/d_0 = 4 \\
 R_1 &= d_1/2 = 1/16 \\
 C_1 &= 1/d_1 = 8 \\
 R_2 &= d_2/6 = 5/48 \\
 C_2 &= 1/d_2 = 8/5
 \end{aligned}$$

Fig. 3.7 presents the final RC circuit structure implementing the given $Z(s)$. The number of capacitors is equal to the degree of the impedance function $Z(s)$. Therefore, it is called the canonical realization.

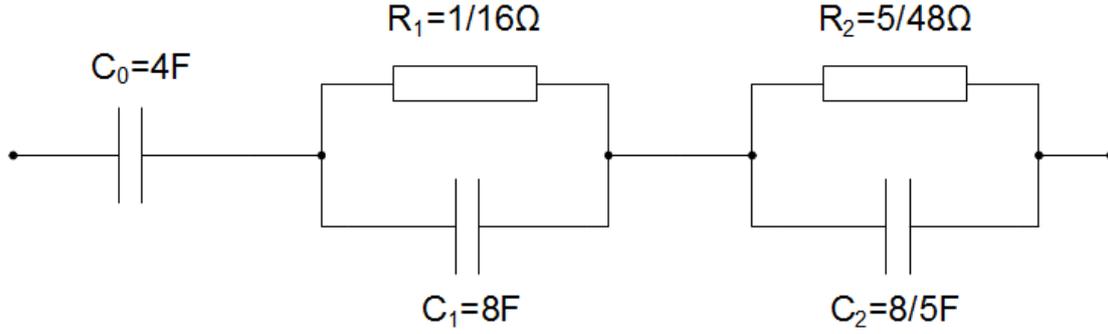


Fig. 3.7 Final RC circuit realizing given $Z(s)$ in the example.

In the case of admittance $Y(s)$ the circuit is composed in the form of the parallel connection of basic RC structures described as follows

$$Y_{RC}(s) = G_0 + sC_0 + \sum_k \frac{s/R_k}{s + \frac{1}{R_k C_k}} \quad (3.23)$$

Hence, the expansion of given $Z(s)$ should take the form

$$Y(s) = a_0 + d_0 s + \sum_k \frac{d_k s}{s - p_k} \quad (3.24)$$

where its elements are described by

$$\begin{aligned} a_0 &= \lim_{s \rightarrow 0} Y(s) \rightarrow G_0 = a_0 \\ d_0 &= \lim_{s \rightarrow \infty} \frac{Y(s)}{s} \rightarrow C_0 = d_0 \\ d_k &= \lim_{s \rightarrow p_k} \frac{Y(s)(s-p_k)}{s} \rightarrow R_k = 1/d_k, C_k = -d_k/p_k \end{aligned} \quad (3.25)$$

Example 3.10

The synthesis procedure will be illustrated on the example of admittance $Y(s)$ given in the form

$$Y(s) = \frac{s^2 + 4s + 3}{(s+2)(s+4)}$$

There are two poles: $p_1=-2$ and $p_2=-4$. Hence the general expansion (3.24) is limited to 4 terms of the following coefficients

$$\begin{aligned} a_0 &= \lim_{s \rightarrow 0} Y(s) = 3/8 \rightarrow G_0 = 3/8 \\ d_0 &= \lim_{s \rightarrow \infty} \frac{Y(s)}{s} = 0 \\ d_1 &= \lim_{s \rightarrow -2} \frac{Y(s)(s+2)}{s} \rightarrow R_1 = 1/d_1 = 4, C_1 = -d_1/p_1 = 1/8 \\ d_2 &= \lim_{s \rightarrow -4} \frac{Y(s)(s+4)}{s} \rightarrow R_2 = 1/d_2 = 8/3, C_2 = -d_2/p_2 = 3/32 \end{aligned}$$

Therefore, the given $Y(s)$ has been expanded to the form

$$Y(s) = \frac{3}{8} + \frac{1/8s}{s+2} + \frac{3/32s}{s+4}$$

Fig. 3.8 presents the final circuit realizing the considered admittance $Y(s)$. It is a parallel connection of the basic RC sections associated with the particular terms of the expansion.

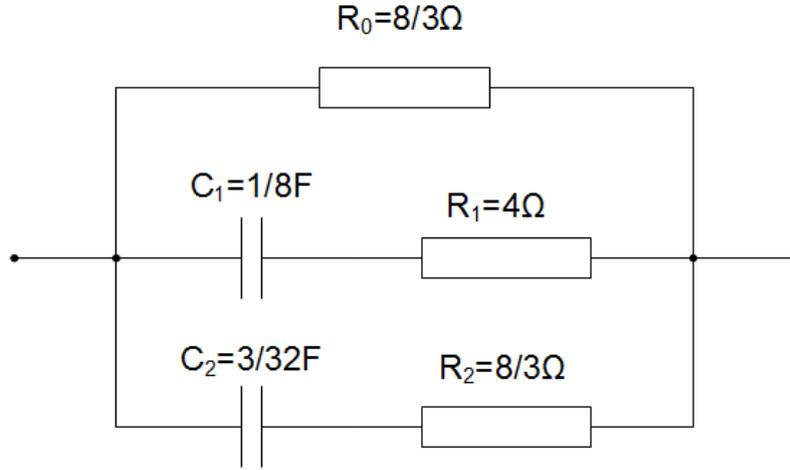


Fig. 3.8 The final form of RC circuit realizing admittance $Y(s)$ in the example.

• **3.3. Foster method in class of LC elements**

The impedance function satisfying the realizability conditions within the class of LC elements can be decomposed as follows

$$Z(s) = a_0 s + d_0/s + \sum_k \frac{d_k s}{s^2 + \omega_k^2} \tag{3.26}$$

where ω_k means the angular frequency on the imaginary axis. Such expansion can be associated with the general LC circuit structure presented in Fig. 3.9, which can be described by the form

$$Z_{LC}(s) = sL_0 + \frac{1}{sC_0} + \sum_k \frac{s/C_k}{s^2 + \frac{1}{L_k C_k}} \tag{3.27}$$

Each term of this expansion (3.26) can be calculated using the following expressions

$$\begin{aligned} a_0 &= \lim_{s \rightarrow \infty} \frac{Z(s)}{s} \\ d_0 &= \lim_{s \rightarrow 0} sZ(s) \\ d_k &= \lim_{s^2 \rightarrow -\omega_k^2} \frac{Z(s)(s^2 + \omega_k^2)}{s} \end{aligned} \tag{3.28}$$

Comparing (3.26) and (3.27) allows identifying the element values of the circuit. We get

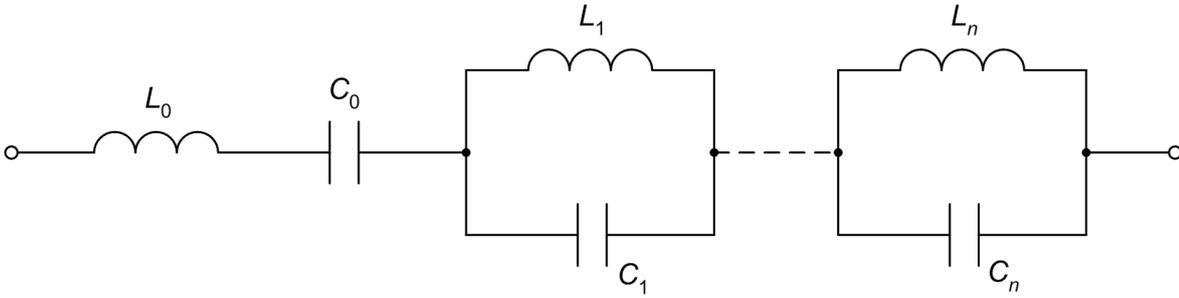


Fig. 3.9 The LC structure realizing impedance $Z(s)$ in Foster method.

$$\begin{aligned}
 L_0 &= a_0 \\
 C_0 &= 1/d_0 \\
 C_k &= 1/d_k \\
 L_k &= d_k/\omega_k^2
 \end{aligned} \tag{3.29}$$

Example 3.11

The procedure of LC synthesis will be illustrated by the exemplary function $Z(s)$

$$Z(s) = \frac{s^6 + 15/4s^4 + 9/4s^2 + 1/4}{s(s^2 + 1)(s^2 + 1/4)}$$

According to Foster procedure we have

$$\begin{aligned}
 a_0 &= \lim_{s \rightarrow \infty} \frac{Z(s)}{s} = 1 \\
 d_0 &= \lim_{s \rightarrow 0} sZ(s) = 1 \\
 d_1 &= \lim_{s^2 \rightarrow -1} \frac{Z(s)(s^2+1)}{s} = 1 \\
 d_2 &= \lim_{s^2 \rightarrow -1/4} \frac{Z(s)(s^2+1/4)}{s} = 1/2
 \end{aligned}$$

It means, that given impedance $Z(s)$ is expanded into the following partial fractions

$$Z(s) = s + \frac{1}{s} + \frac{s}{s^2 + 1} + \frac{1/2s}{s^2 + 1/4}$$

From (3.29) we identify the element values:

$$\begin{aligned}
 L_0 &= a_0 = 1 \\
 C_0 &= 1/d_0 = 1 \\
 C_1 &= 1/d_1 = 1 \\
 L_1 &= d_1/\omega_1^2 = 1 \\
 C_2 &= 1/d_2 = 2 \\
 L_2 &= d_2/\omega_2^2 = 2
 \end{aligned}$$

Fig. 3.10 shows the final structure of LC circuit realizing given $Z(s)$. This is also the canonical realization (number of L and C elements is the same as the degree of the synthesized function $Z(s)$).

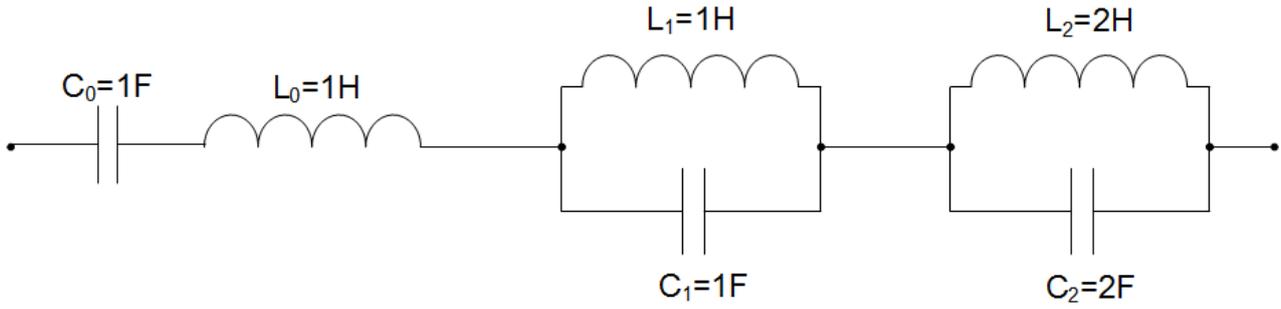


Fig. 3.10 The LC circuit structure corresponding to the given $Z(s)$.

In a similar way we can complete the synthesis procedure of LC circuit in the case of given admittance function. The $Y_{LC}(s)$ is the parallel connection of series sections presented in Fig. 3.3, of the admittance form given as follows

$$Y_{LC}(s) = sC_0 + \frac{1}{sL_0} + \sum_k \frac{s/L_k}{s^2 + \frac{1}{L_k C_k}} \tag{3.30}$$

Hence, the expected form of expansion is as follows

$$Y(s) = a_0 s + \frac{d_0}{s} + \sum_k \frac{d_k s}{s^2 + \omega_k^2} \tag{3.31}$$

From (3.28) and (3.30) we get

$$\begin{aligned} a_0 &= \lim_{s \rightarrow \infty} \frac{Y(s)}{s} \rightarrow C_0 = a_0 \\ d_0 &= \lim_{s \rightarrow 0} sY(s) \rightarrow L_0 = 1/d_0 \\ d_k &= \lim_{s \rightarrow -\omega_k^2} \frac{Y(s)(s^2 + \omega_k^2)}{s} \rightarrow L_k = 1/d_k, C_k = d_k/\omega_k^2 \end{aligned}$$

3.5. Cauer method of synthesis of RLC circuits

Cauer method of synthesis is a bit more general, since in some cases it allows to deal with 3-element circuits containing R, L and C. The method is based on expansion of the given immittance into continued fraction, which can be next associated with ladder form of basic RLC elements.

A **continued fraction** is an expression obtained through an iterative process of representing a number as the sum of its integer part and the reciprocal of another number, then writing this other number as the sum of its integer part and another reciprocal, and so on.

The condition of realizability in ladder form is that all terms of continued fractions should correspond to the basic descriptions of particular passive elements: $ZL=sL$ or $YL=1/sL$ for inductors, $ZC=1/sC$ or $YC=sC$ for capacitors and $ZR=R$ or $YR=G$ for resistors.

Two different forms of **ladder network** can be recognized. The first one corresponds to the structure presented in Fig. 3.11b. The continued fraction corresponding to it is given in the form [1,16]

$$Z(s) = Z_1 + \frac{1}{Y_2 + \frac{1}{Z_3 + \frac{1}{Y_4 + \dots}}} \tag{3.32}$$

It can be obtained by dividing the polynomial of numerator by the polynomial of denominator. In the next steps we inverse both polynomials of the rest and continue the process of division.

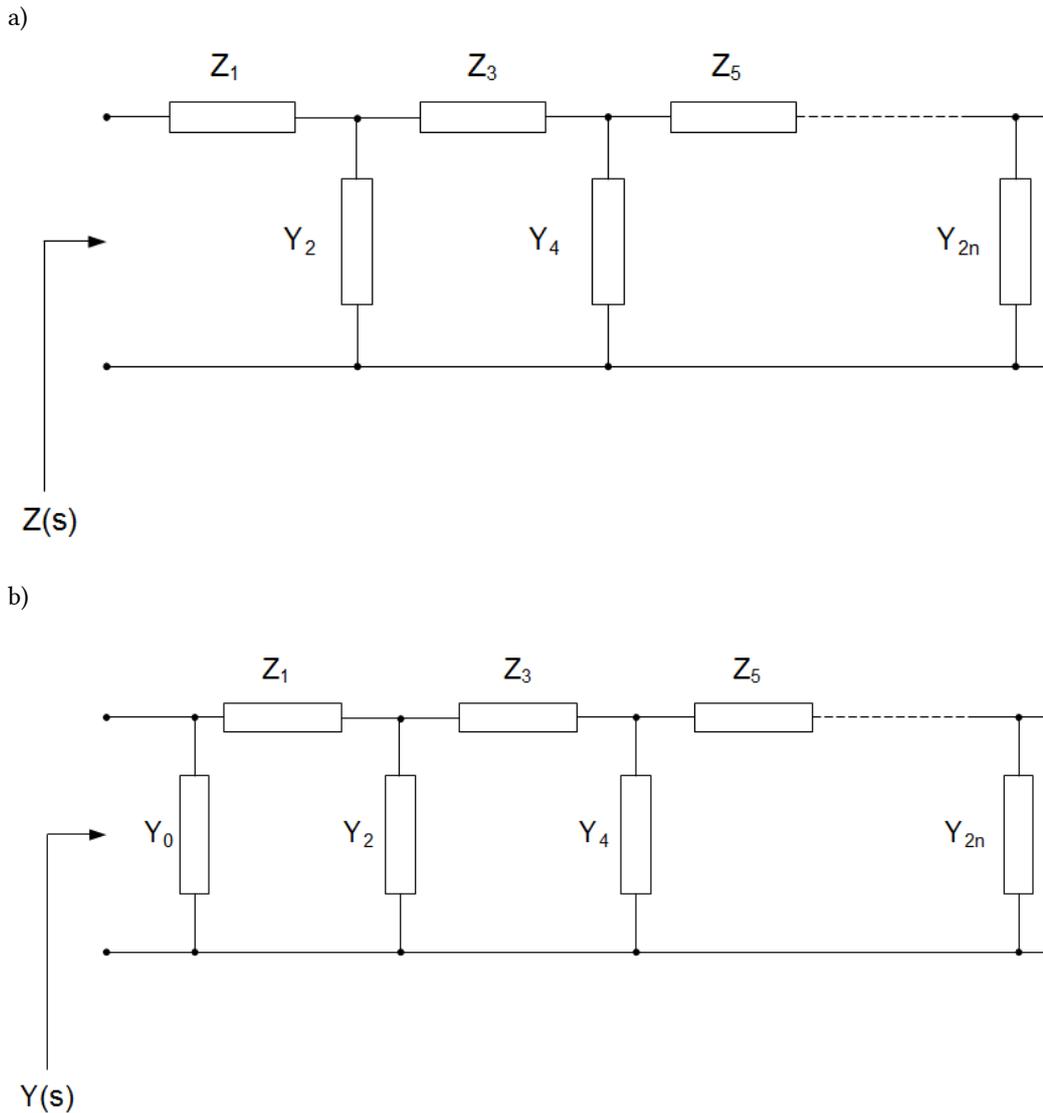


Fig. 3.11 Two canonical forms of ladder networks realizing: a) the impedance $Z(s)$ and b) the admittance $Y(s)$.

Performing the same operations on the admittance we get the alike form for the admittance, corresponding to the circuit structure of Fig. 3.11b [1,16]

$$Y(s) = Y_0 + \frac{1}{Z_1 + \frac{1}{Y_2 + \frac{1}{Z_3 + \dots}}} \quad (3.33)$$

The process of division should be organized in a way to get all elementary elements ($Z_1, Y_2, Z_3, Y_4 \dots$) of the positive numerical coefficients and of the form corresponding to the allowed passive elements: pure numerical value for resistor, $Z_L = sL$ for inductor and $Z_C = 1/sC$ for capacitor. The process of division may be performed on the highest or the lowest powers of both polynomials. The only criterion is to get the final result satisfying the realizability conditions in the ladder form.

Example 3.12

The procedure of Cauer synthesis will be illustrated on the example below

$$Z(s) = \frac{s^4 + 3s^3 + 4s^2 + 4s + 1}{2s^3 + 2s^2 + 3s + 1}$$

In the first step we divide both polynomials according to the lowest power of polynomials. Hence, we get

$$Z(s) = 1 + \frac{s^4 + s^3 + 2s^2 + s}{2s^3 + 2s^2 + 3s + 1} = 1 + \frac{1}{\frac{2s^3 + 2s^2 + 3s + 1}{s^4 + s^3 + 2s^2 + s}} = 1 + \frac{1}{Y_{2r}}$$

In the next step we continue division of the inversed rest (also according to the lowest powers). As a result we get

$$Y_{2r} = 1/s + \frac{s^3 + s^2 + s}{s^4 + s^3 + 2s^2 + s} = 1/s + \frac{1}{\frac{s^4 + s^3 + 2s^2 + s}{s^3 + s^2 + s}} = 1/s + \frac{1}{Z_{3r}}$$

In the next step we continue the same process on the inversed rest Z_{3r} , dividing the polynomials according to the highest powers of polynomials. Hence, we get

$$Z_{3r} = s + \frac{s^2 + s}{s^3 + s^2 + s} = s + \frac{1}{\frac{s^3 + s^2 + s}{s^2 + s}} = s + \frac{1}{Y_{4r}}$$

The next steps are performed on the inversed rests and all divisions are done according to the highest powers. As a result we get

$$Y_{4r} = s + \frac{s}{s^2 + s} = s + \frac{1}{\frac{s^2 + s}{s}} = s + \frac{1}{Z_{5r}}$$

$$Z_{5r} = s + \frac{s}{s} = s + \frac{1}{1}$$

At the end we get the final form of continued fraction

$$Z(s) = 1 + \frac{1}{\frac{1}{1/s + \frac{1}{s + \frac{1}{s + \frac{1}{s + \frac{1}{1}}}}}}$$

Comparing it with the general form described by (3.32) we can identify the succeeding elements of the ladder network. They are: $Z_1=1$, $Y_2=1/s$, $Z_3=s$, $Y_4=s$, $Z_5=s$, $Y_6=1$. Hence the final form of ladder implementing given $Z(s)$ is shown in Fig. 3.12.

It should be observed, that the process of division of polynomials is in general not unique. The only hint, how to organize it, is the final results, which should satisfy the realizability conditions of ladder using passive elements RLC. Only in the case of 2-class circuits the process of division is unique and repeatable in all stages (division either according to the highest or to the lowest powers).

3.6. Brune method of RLC synthesis

Brune method is applicable to a bit more general RLC circuit, since it applies also mutually couples inductances. The procedure of synthesis of $Z(s)$ in Brune method can be presented in the following way [1,16].

1. Extract all poles or zeros located on imaginary axis (if exist). It is done according to the Foster method for LC elements. After their extraction the impedance $Z(s)$ is simplified to $Z_1(s)$.

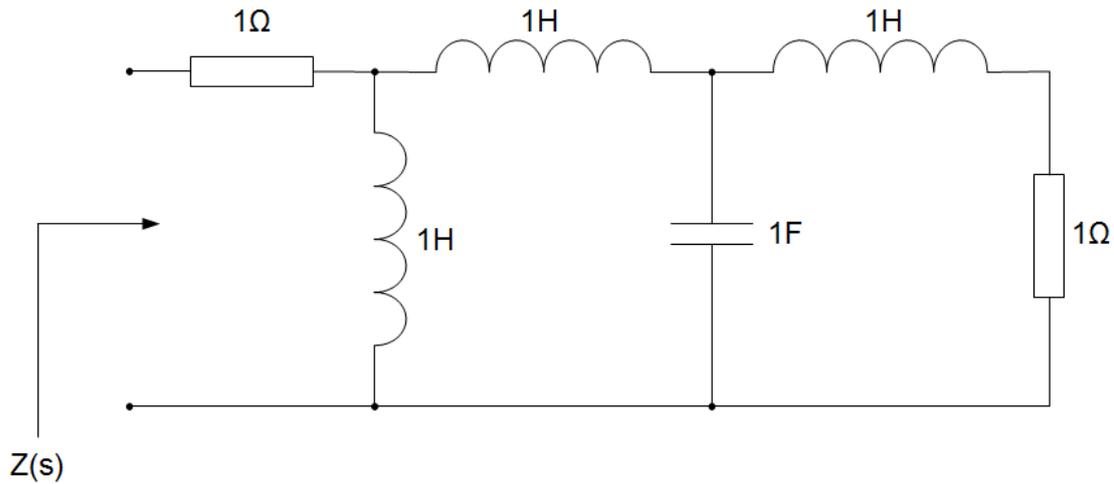


Fig. 3.12 Ladder network form implementing the impedance $Z(s)$ of the example.

2. Determine the angular frequency ω_0 for which $\text{Re}[Z_1(j\omega)]$ takes the minimum value. Then we extract the resistance R_1 equal $R_1 = \text{Re}[Z_1(j\omega_0)]$. As a result we get $Z_2(s) = Z_1(s) - R_1$ and further process of synthesis is done on this function.
3. If R_1 corresponds to $\omega_0=0$ or $\omega_0=\infty$ the impedance $Z_2(s)$ is realizable in the class of two elements and can be completed using Foster or Caueer method.
4. On the other side, if R_1 corresponds to other value of ω_0 we have to calculate $Z_2(j\omega_0) = jX_2$ and extract the inductance $L_2 = X_2/\omega_0$ (the value of L_2 might be negative). After extraction of L_2 the impedance is simplified to $Z_3(s) = Z_2(s) - sL_2$.
5. For such impedance $Z_3(s)$ the procedure starts from the first step until the function is reduced to zero.
6. The final circuit is the series connection of extracted elements (if the impedances were under considerations) or parallel connection of elements in the case of admittances. If the extracted inductance is negative we have to substitute it by proper arrangement of magnetically coupled inductive elements.

Example 3.13

As an illustrative example let us take the following impedance $Z(s)$

$$Z(s) = \frac{s^2 + s + 1}{s^2 + s + 2}$$

- According to the first step of synthesis we see that there are no poles and no zeros on the imaginary axis. Therefore, $Z_1(s) = Z(s)$.
- The second step requires to determine $\text{Re}[(Z_1(j\omega))]$. After substituting $s = j\omega$ to $Z_1(s)$ we get

$$\text{Re}[Z_1(j\omega)] = \text{Re} \left[\frac{(1 - \omega^2) + j\omega}{(2 - \omega^2) + j\omega} \right] = \frac{\omega^4 - 2\omega^2 + 2}{\omega^4 - 3\omega^2 + 4}$$

The minimum of the function requires derivative equal zero. Therefore,

$$\frac{d}{d\omega} \text{Re}\{Z_1(j\omega)\} = 0 \rightarrow \frac{d}{d\omega} \left(\frac{\omega^4 - 4\omega^2 + 2}{\omega^4 - 3\omega^2 + 2} \right) = 0$$

After simplification of this function we get

$$\omega(\omega^4 - 4\omega^2 + 2) = 0$$

Solving it, we obtain 3 values of solution

$$\begin{aligned} \omega_1 = 1.85 &\rightarrow \text{Re}[Z_1(j1.85)] = 1.25 \\ \omega_2 = 0.7655 &\rightarrow \text{Re}[Z_1(j0.7655)] = 0.453 \\ \omega_3 = 0 &\rightarrow \text{Re}[Z_1(j0)] = 0, 5 \end{aligned}$$

The value ω_0 should correspond to the minimum of real part of $Z_1(j\omega)$. Hence we have $\omega_0 = 0.7655$.

Extracting the resistance $R_1 = \text{Re}[Z_1(j\omega_0)] = 0.453$ we get

$$Z_2(s) = Z_1(s) - R_1 = \frac{0.547s^2 + 0.547s + 0.094}{s^2 + s + 2}$$

- Because the minimum corresponds to the finite frequency we move to the fourth stage of the procedure. According to it we have $Z_2(j\omega_0) = jX_2$ and

$$Z_2(j\omega_0) = Z_2(j0.7655) = j0.296$$

It corresponds to $X_2 = \omega_0 L_2 = 0.296$. Hence $L_2 = 0.296 / 0.7655 = 0.387$. After extracting inductance L_2 we get

$$Z_3(s) = Z_2(s) - sL_2 = \frac{(s^2 + 0.586)(-0.387s + 0.16)}{s^2 + s + 2}$$

- The function $Z_3(s)$ has 2 zeros on the imaginary axis $s_2 = -0.586$. Hence further procedure will be performed on the admittance $Y_3(s) = 1/Z_3(s)$ (synthesis is always based on the poles and not on the zeros)

$$Y_3(s) = \frac{s^2 + s + 2}{(s^2 + 0.586)(-0.387s + 0.16)}$$

- Extraction of poles $s_2 = -0.586$ is based on Foster method for LC circuit. Hence

$$d_3 = \lim_{s^2 \rightarrow -0.586} \frac{Y_3(s)(s^2 + 0.586)}{s} = 6.25 \rightarrow L_3 = 1/d_3 = 0.16, C_3 = 6.25/0.586 = 10.66$$

$$Y_{L_3 C_3}(s) = \frac{6.25s}{s^2 + 0.586}$$

The defined LC section is described now as $Y_3(s)$ we get $Y_4(s)$, where

$$Y_4(s) = Y_3 - \frac{6.25s}{s^2 + 0.586} = \frac{1}{-0.113s + 0.047}$$

It corresponds to the impedance $Z_4(s) = 1/Y_4(s)$. Hence, $Z_4 = -0.113s + 0.047$, which means series connection of resistance $R_4 = 0.047$ and inductive impedance $Z_{L4} = -0.113s$ (negative inductance $L_4 = -0.113$). Fig. 3.13 presents

the final form of circuit resulting from the synthesis method. Fig. 3.13a operates with the original negative inductances and Fig. 3.13b shows the practical implementation of these negative inductances by applying mutually coupled inductors (all circuit elements are now positive).

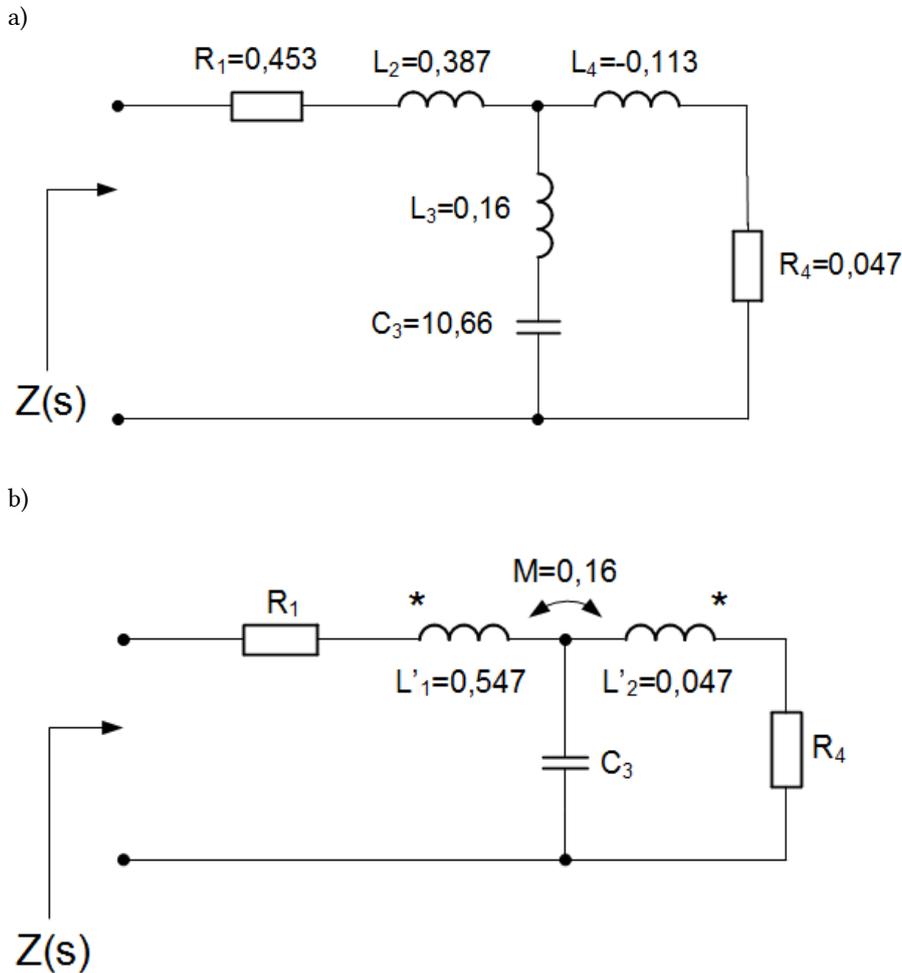


Fig. 3.13 The circuit structure implementing the given impedance $Z(s)$ of the example: a) the original structure with negative inductances and b) the final form of circuit with magnetically coupled positive inductances.

Example 3.14

In the next example we consider the impedance function of the third order, given as follows

$$Z(s) = \frac{s(s^2 + s + 2)}{s^3 + 3s^2 + 2s + 2}$$

- This function possesses one zero on the imaginary axis $s=0$. We can extract it as a pole of admittance $Y(s)$

$$Y(s) = \frac{s^3 + 3s^2 + 2s + 2}{s(s^2 + s + 2)}$$

Applying Foster method for LC circuit we get

$$d_0 = \lim_{s \rightarrow 0} sY(s) = 1 \rightarrow L_0 = 1/d_0 = 1 \rightarrow Y_0 = 1/sL_0 = 1/s$$

After extraction of this inductance we get

$$Y_1(s) = Y(s) - \frac{1}{s} = \frac{(s + 1)^2}{s^2 + s + 2}$$

- In the second step we have to determine $\text{Re}[Y_1(j\omega)]$. After substitution of $s=j\omega$ to the admittance $Y_1(s)$ we get

$$\text{Re}[Y_1(j\omega)] = \frac{\omega^4 - \omega^2 + 2}{(2 - \omega^2)^2 + \omega^2}$$

The minimum of the function requires

$$\frac{d}{d\omega} \text{Re}[Y_1(j\omega)] = 0 \rightarrow \frac{-4\omega^3 + 8\omega}{(2 - \omega^2)^2 + \omega^2} = 0$$

Solving it we get three values of frequency, from which one is negative (not valid in further procedure)

$$\begin{aligned} \omega_1 = 0 &\rightarrow \text{Re}[Y_1(j0)] = 0,5 \\ \omega_2^2 = 2,41 &\rightarrow \omega_2 = 1,55 \rightarrow \text{Re}[Y_1(j1,55)] = 2,09 \\ \omega_3^2 = -0,41 &\rightarrow - \end{aligned}$$

Choose $\omega_0=0$, for which the real part of admittance takes minimum value. Extracting $G_1 = \text{Re}[Y_1(j\omega_0)] = 0,5$ we get

$$Y_2(s) = Y_1(s) - G_1 = \frac{s(s + 3)}{2s^2 + 2s + 4}$$

- Because the minimum has occurred at zero, further synthesis procedure will be done according to Foster method on the basis of $Z_2(s)=1/Y_2(s)$

$$Z_2(s) = \frac{1}{Y_2(s)} = \frac{2s^2 + 2s + 4}{s(s + 3)}$$

This impedance has two poles $p_1=0$ and $p_2=-3$. After extracting the first pole we get

$$d_1 = \frac{1}{C_1} = \lim_{s \rightarrow 0} sZ_2(s) = \frac{4}{3} \rightarrow C_1 = \frac{3}{4}$$

The extracted impedance is equal $Z_{C_1} = \frac{4}{3s}$. The residual impedance $Z_3(s)$ is now described in the form

$$Z_3(s) = Z_2(s) - \frac{4}{3s} = \frac{2s}{s + 3} + \frac{2/3}{s + 3}$$

- The first term of this impedance function corresponds to the parallel connection of resistor and inductor. Hence,

$$Z_{RL} = \frac{sR_2}{s + R_2/L_2} = \frac{2s}{s + 3} \rightarrow R_2 = 2, L_2 = 2/3$$

The second term describes the parallel connection of resistor and capacitor. Hence,

$$Z_{RC} = \frac{1/C_3}{s + 1/R_3 C_3} = \frac{2/3}{s + 3} \rightarrow R_3 = 2/9, C_3 = 3/2$$

Fig. 3.14 presents the final form of Brune circuit in implementation of the given function $Z(s)$ of the example.

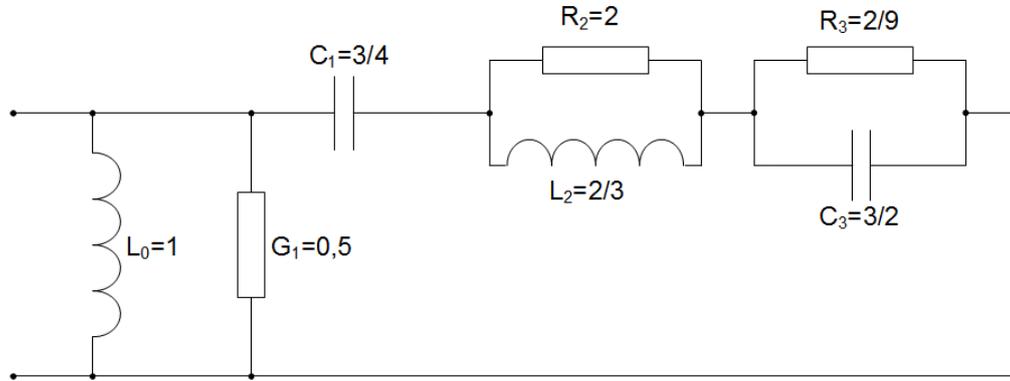


Fig. 3.14 The structure of Brune circuit realizing impedance $Z(s)$

3.7. Darlington method of synthesis of RLC circuit

Darlington method is based on the lossless 2-port loaded on output port by the resistor R . The general scheme of Darlington structure is shown in Fig. 3.15.

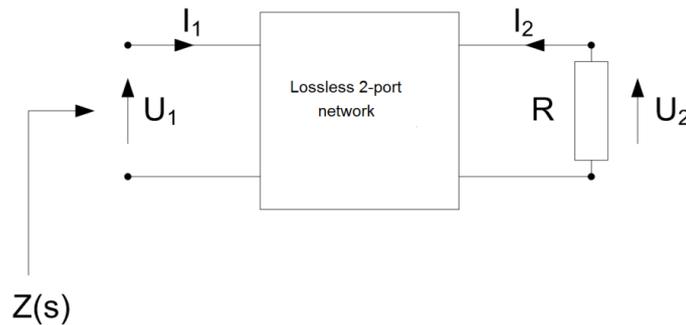


Fig. 3.15 Darlington structure used in synthesis.

The procedure assumes the impedance description of the 2-port network by the matrix \mathbf{Z} .

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

(3.34)

Input impedance of the 2-port loaded by resistor R can be expressed in the form

$$Z(s) = \frac{U_1}{I_1} = Z_{11} - \frac{Z_{12}Z_{21}}{Z_{22} + R} = Z_{11} \frac{\left(Z_{22} - \frac{Z_{12}Z_{21}}{Z_{11}} \right) + R}{Z_{22} + R} \quad (3.35)$$

The given impedance $Z(s)$ should be transformed to the form, which allows identifying the impedance matrix parameters defined by (3.35).

Let us introduce the even and odd parts of the impedance $Z(s)$ as Mp , Lp (even polynomials of denominator

and numerator, respectively), and M_n, L_n (odd parts of the corresponding polynomials). In such case the impedance function $Z(s)$ may be presented in the form

$$Z(s) = \frac{L(s)}{M(s)} = \frac{L_p + L_n}{M_p + M_n} = \frac{L_p}{M_n} \frac{\frac{L_n}{L_p} + 1}{\frac{M_p}{M_n} + 1} \quad (3.36)$$

Assuming $R=1$, from the comparison of (3.35) and (3.36), we can identify the particular element of the impedance matrix \mathbf{Z} .

$$\begin{aligned} Z_{11} &= \frac{L_p}{M_n} \\ Z_{22} &= \frac{M_p}{M_n} \\ Z_{22} - \frac{Z_{12}Z_{21}}{Z_{11}} &= \frac{L_n}{L_p} \rightarrow Z_{12}Z_{21} = \frac{L_pM_p - L_nM_n}{M_n^2} \end{aligned} \quad (3.37)$$

In identification of Z_{12} and Z_{21} parameters the significant is the reciprocity (invertibility) of the 2-port. The 2-port is reciprocal if $Z_{12}=Z_{21}$. At satisfaction of this condition we get

$$Z_{12} = \frac{\sqrt{L_pM_p - L_nM_n}}{M_n} \quad (3.38)$$

It means that the term $L_pM_p - L_nM_n$ should be expressed as the squared polynomial.

The additional problem is the value of resistance R (up to now assumed as unity). The equation (3.36) can be scaled by an arbitrary value of R . After scaling, the parameter Z_{11} will not be changed. The parameter Z_{22} will be scaled by R while the parameters Z_{12} and Z_{21} will be scaled by \sqrt{R} . After such scaling we get the transformed values of parameters of \mathbf{Z} matrix.

$$\begin{aligned} Z'_{11} &= Z_{11} \\ Z'_{22} &= R \cdot Z_{22} \\ Z'_{12} &= \sqrt{R} \cdot Z_{12} \end{aligned} \quad (3.39)$$

The matrix \mathbf{Z} can be now expressed in the form

$$\mathbf{Z} = \begin{bmatrix} \frac{L_p}{M_n} & \sqrt{R} \frac{(P_p + P_n)}{M_n} \\ \sqrt{R} \frac{(P_p - P_n)}{M_n} & R \frac{M_p}{M_n} \end{bmatrix} \quad (3.40)$$

This matrix will be implemented in a proper circuit form using RLC elements.

The reciprocity condition of 2-port network is not always possible. In such case we can introduce the additional passive element in the form of gyrator, which represents the non-reciprocal 2-port. It is described by the following matrix equation

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 & \pm R_z \\ \mp R_z & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad (3.41)$$

If the 2-port is not reciprocal the product $Z_{12}Z_{21}$ can be expressed in the form

$$Z_{12}Z_{21} = \frac{L_pM_p - L_nM_n}{M_n^2} = \frac{(P_p + P_n)(P_p - P_n)}{M_n^2} \quad (3.42)$$

The terms P_p and P_n represent the polynomials of the even powers (P_p) or odd powers (P_n). They should be adjusted in a way to satisfy both sides of eq. (3.42). From the comparison of both sides we get

$$\begin{aligned} Z_{12} &= \frac{(P_p + P_n)}{M_n} \\ Z_{21} &= \frac{(P_p - P_n)}{M_n} \end{aligned} \quad (3.43)$$

After this substitution we get the impedance matrix \mathbf{Z} in the form

$$\mathbf{Z} = \begin{bmatrix} \frac{L_p}{M_n} & \sqrt{R} \frac{(P_p + P_n)}{M_n} \\ \sqrt{R} \frac{(P_p - P_n)}{M_n} & R \frac{M_p}{M_n} \end{bmatrix} \quad (3.44)$$

The general procedure of implementation the \mathbf{Z} matrix in a circuit form will require the application of RLC elements, magnetically coupled coils and gyrator. The computation of the values of these elements needs decomposition of the matrix elements into the elementary submatrices. Due to the impedance matrix description they correspond to the series connection of the appropriate 2-ports. It will be illustrated on the following example.

Example 3.14

Let us take the impedance $Z_{in}(s)$ in the form

$$Z_{in}(s) = \frac{2s^2 + 5s + 2}{0.5s^2 + 2s + 1}$$

From the given form of $Z(s)$ it follows

$$\begin{aligned} L_p &= 2s^2 + 2 \\ L_n &= 5s \\ M_p &= 0.5s^2 + 1 \\ M_n &= 2s \end{aligned}$$

At $R=1$ we get

$$\begin{aligned} Z_{11} &= \frac{L_p}{M_n} = \frac{2s^2 + 2}{2s} = s + 1/s \\ Z_{22} &= \frac{M_p}{M_n} = \frac{0.5s^2 + 1}{2s} = 0.25s + 0.5/s \end{aligned}$$

The expression $Z_{12}Z_{21} = L_p M_p - L_n M_n$ does not allow to be presented as the squared polynomial. Hence the 2-port is not reciprocal and needs application of gyrator. According to the procedure for non-reciprocal 2-port we introduce even and odd polynomials defined in the way

$$L_p M_p - L_n M_n = (P_p + P_n)(P_p - P_n)$$

The second order polynomial $P(s) = P_p(s) + P_n(s)$ is needed. Hence, $P_p = s^2 + a_0$, $P_n = a_1 s$. From the comparison of both sides of the last equation we get

$$(2s^2 + 2)(0.5s^2 + 1) - 10s^2 = (s^2 + a_1 s + a_0)(s^2 - a_1 s + a_0)$$

Hence,

$$\begin{aligned} -7s^2 &= s^2(2a_0 - a_1^2) \rightarrow a_1 = 3.12 \\ 2 &= a_0^2 \rightarrow a_0 = 1.41 \end{aligned}$$

As a result the polynomials P_p i P_n are of the form

$$P_p = s^2 + 1.41$$

$$P_n = 3.12s$$

The parameters Z_{12} and Z_{21} are then equal

$$Z_{12} = \frac{(P_p + P_n)}{M_n} = \frac{s^2 + 3.12s + 1.41}{2s} = 0.5s + 1.56 + \frac{0.705}{s}$$

$$Z_{21} = \frac{(P_p - P_n)}{M_n} = \frac{s^2 - 3.12s + 1.41}{2s} = 0.5s - 1.56 + \frac{0.705}{s}$$

The impedance matrix \mathbf{Z} is expressed in the form

$$\mathbf{Z} = \begin{bmatrix} s + 1/s & 0.5s + 1.56 + 0.705/s \\ 0.5s - 1.56 + 0.705/s & 0.25s + 0.5/s \end{bmatrix}$$

which can be presented as the sum of the following submatrices

$$\mathbf{Z} = s \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.25 \end{bmatrix} + \frac{1}{s} \begin{bmatrix} 1 & 0.705 \\ 0.705 & 0.5 \end{bmatrix} + \begin{bmatrix} 0 & 1.56 \\ -1.56 & 0 \end{bmatrix} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3$$

All of them represent particular 2-ports, connected in series. The first term (matrix \mathbf{Z}_1) represents the magnetically coupled inductors. The second one (matrix \mathbf{Z}_2) should be realized by using capacitors. The third term (matrix \mathbf{Z}_3) represents **gyrator**.

The basic limitations are visible in realization of the second term, since the capacitance matrix should be

$$\mathbf{Z}_2 = \frac{1}{s} \begin{bmatrix} 1 & 0.705 \\ 0.705 & 0.5 \end{bmatrix}$$

hyperdominant. In its present form

it is not. To provide the **hyperdominance**

condition the resistance scaling is necessary.

The scaling procedure changes the elements of \mathbf{Z} matrix in the following way

$$Z'_{11} = Z_{11}$$

$$Z'_{22} = R \cdot Z_{22}$$

$$Z'_{12} = \sqrt{R} \cdot Z_{12}$$

The resistance R should be adjusted in such way, that $Z'_{22} = Z'_{21}$ (the parameters of matrix \mathbf{Z}_2), hence,

$$R = \left(\frac{0.705}{0.5} \right)^2 = 2$$

. However, this scaling should be done for all submatrices: \mathbf{Z}_1 , \mathbf{Z}_2 and \mathbf{Z}_3 . Applying $R=2$ we obtain the matrix \mathbf{Z} in the form, satisfying the hyperdominance conditions for \mathbf{Z}_2

$$\mathbf{Z} = s \begin{bmatrix} 1 & 0.705 \\ 0.705 & 0.5 \end{bmatrix} + \frac{1}{s} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2.2 \\ -2.2 & 0 \end{bmatrix}$$

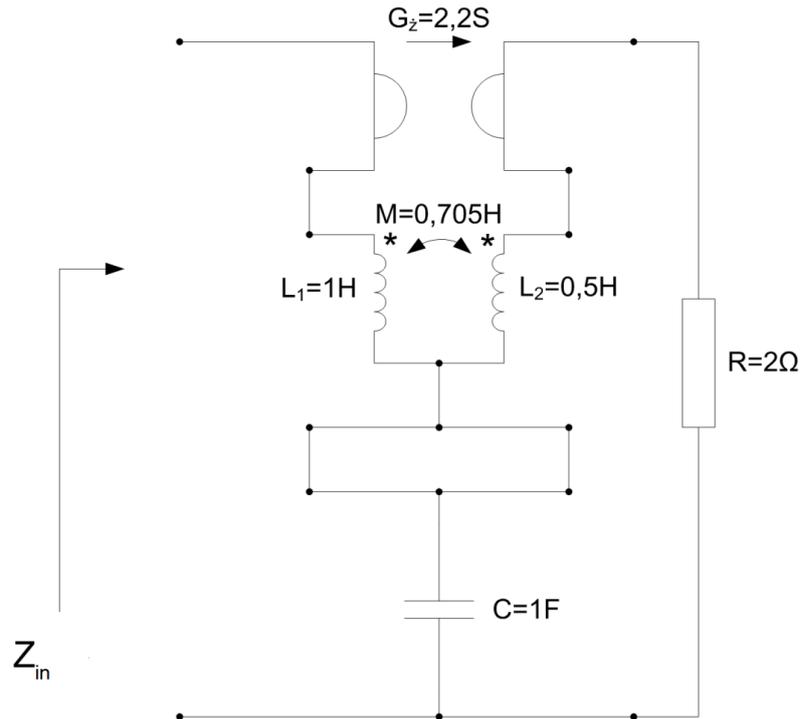
Its circuit implementation is presented in Fig. 3.16. It is formed by three 2-ports connected in series.

3.8 Exercises

Check realizability conditions of given function $Z(s)$ using RL or RC elements:

a)
$$Z(s) = \frac{(s + 2)(s + 4)}{(s + 1)(s + 3)(s + 5)}$$

Solution:



Rys 3.16

Fig. 3.16 Final form of circuit realizing the given impedance $Z_{we}(s)$ in Darlington scheme.

The realizability conditions in RC circuit are met, since the coefficients of numerator and denominator are positive, the poles and zeros are real and alternate and closest to the origin is the pole of $Z(s)$ $s=-1$.

$$b) \quad Y(s) = \frac{s(s+3)}{(s+2)(s+4)(s+5)}$$

Solution:

The function is not realizable in RC nor RL two element circuit, since the poles and zeros do not alternate. The zeros are: $z_1=0$, $z_2=-3$. Poles: $s_1=-2$, $s_2=-4$, $s_3=-5$. Poles -4 and -5 are not separated by any zero.

$$Z(s) = \frac{s^2 + 6s + 5}{s^2 + 3s}$$

Synthesize function

Solution:

The impedance $Z(s)$ may be presented in zero-pole form as follows

$$Z(s) = \frac{(s+1)(s+5)}{s(s+3)}$$

where zeros are: -1, -5 and poles: 0, -3. It is evident that this function satisfies the realizability condition using RC elements. Therefore the expected first order expansion of this function may be presented in the form

$$Z(s) = a + \frac{b}{s} + \frac{d}{(s+3)}$$

where

$$a = \lim_{s \rightarrow \infty} Z(s) = 1$$

$$b = \lim_{s \rightarrow 0} s \cdot Z(s) = 5/3$$

$$d = \lim_{s \rightarrow -3} (s + 3) \cdot Z(s) = 4/3$$

It is evident now that $R_0=a=1\Omega$, $C_0=1/b=3/5F$, $C_1=1/d=3/4F$, $R_1=1/(3C_1)=4/9\Omega$. The final circuit structure is presented in figure below.

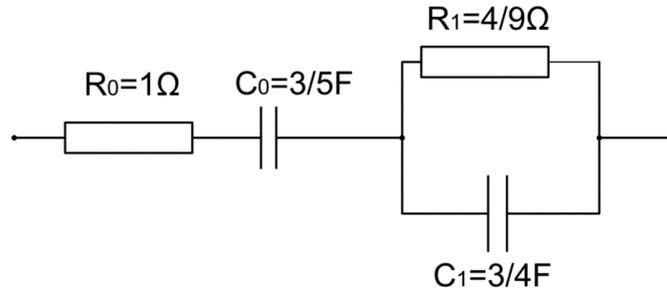


Fig. 3.17 Foster circuit realizing the given impedance $Z(s)$

$$Z(s) = \frac{s^4 + 3s^2 + 1}{s^3 + s}$$

Synthesize function using Foster method.

Solution:

The impedance $Z(s)$ may be presented in zero-pole form as follows

$$Z(s) = \frac{(s^2 + 2.618)(s^2 + 0.3821)}{s(s^2 + 1)}$$

The function is realizable using LC elements, since the poles and zeros are on the imaginary axis and alternate. Therefore the expected expansion of this function may be presented in the form

$$Z(s) = as + \frac{b}{s} + \frac{ds}{(s^2 + 1)}$$

where

$$a = \lim_{s \rightarrow \infty} \frac{Z(s)}{s} = 1$$

$$b = \lim_{s \rightarrow 0} s \cdot Z(s) = 1$$

$$d = \lim_{s \rightarrow -1} \frac{Z(s)}{s} (s^2 + 1) = 1$$

Therefore we get

$$L_0=a=1H, \quad C_0=1/b=1F, \quad C_1=1/d=1F, \quad L_1=1/C_1=1H$$

The LC circuit structure implementing the given impedance function $Z(s)$ is presented in figure below.

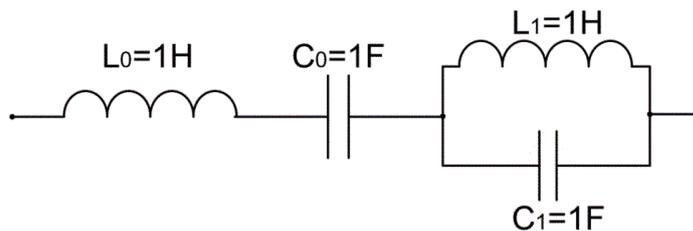


Fig. 3.18 Foster LC circuit implementing given $Z(s)$.

1. Decompose $Z(s)$ of ladder network into continued fraction form

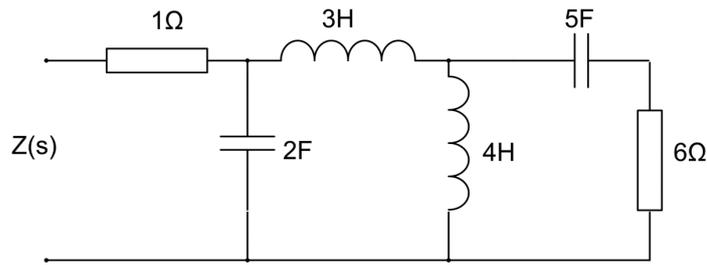


Fig. 3.19 The ladder RLC circuit

Solution:

Dividing numerator by denominator and inverting the residue we obtain finally the continued fraction form of $Z(s)$

$$Z(s) = 1 + \frac{1}{2s + \frac{1}{3s + \frac{1}{1/4s + \frac{1}{6 + 1/5s}}}}$$

1. Apply Cauer method to realize given impedance function $Z(s)$

$$Z(s) = \frac{2s^2 + 5s + 4}{2s^2 + 3s + 1}$$

Solution:

Dividing the polynomials of numerator and denominators by the highest order we get

$$Z(s) = 1 + \frac{1}{s + \frac{1}{2s+3}}$$

This continued fraction corresponds to the ladder network presented in figure below.

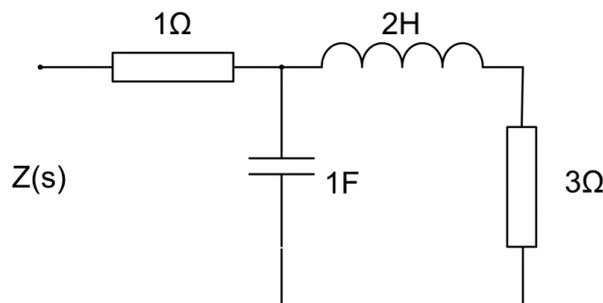


Fig. 3.20 Cauer ladder circuit structure implementing given $Z(s)$

3.9 Basic definitions

Active element – element representing source of energy such as independent and controlled sources.

Brune method – method of synthesis of RLC circuits containing mutually coupled inductors.

Canonical section – the parallel or series connection of two types of elements: RL, RC or LC, which are used in Foster method.

Cauer method – method of synthesis of RLC circuits based on expansion of the given immittance into **continued fraction**. The resulting expansion is transformed into a ladder network of basic RLC elements.

Continued fraction – the **expression** obtained through an **iterative** process of representing a rational function of two polynomials as the sum of the result of their division according to the highest or lowest power and the **reciprocal** of the rest, then repeating the process on the reciprocal of this rest, and so on.

Darlington method – method of synthesis of RLC circuits containing mutually coupled inductors and gyrators.

Foster method – method of synthesis based on decomposition of given immittance into the sum of the first order terms (partial fractions) and association of these terms with particular canonical sections containing RL, RC or LC elements.

Gyrator – a **passive**, linear, lossless, **two-port** electrical **network element** It is **non-reciprocal** 2-port. It inverts the **current-voltage characteristic** of an **electrical component** or **network**. The **impedance** is also inverted. In other words, a gyrator can make a **capacitive** circuit behave **inductively**, a **series LC circuit** behave like a **parallel LC circuit**, and so on.

Hyperdominant matrix – matrix satisfying condition that the sum of the absolute values of off-diagonal element in each row is smaller than the diagonal element value.

Impedance matrix – the matrix describing the behavior of any linear electrical network (for example 2-port) that can be regarded as a **black box** with a number of **ports** (pair of **electrical terminals**). On the left side of description we have port voltages and on the right – the port currents.

Ladder network – the string of many electrical elements (resistors, capacitors, inductors) connected in a structure resembling ladder.

Passive element – the element which can't produce energy (increase the power of a signal). Passive components include two-terminal components such as resistors, capacitors, inductors and four-terminal components such as transformers or gyrators.

Realizability conditions – the collection of conditions that should be satisfied in converting the given immittance function into the circuit structure realizable in practice.

Synthesis – the process of converting a given transfer function form into a particular circuit structure implementation.

3. Electrical filters

The important task in engineering practice is frequency filtration, performed by the devices called filters. The task of filter is to pass or stop signals within defined frequency range. Filters might be built in analog or digital implementation. This chapter will consider only the analog technology. In such approach filter is described by the rational transfer function $H(s)$, where s represents the complex frequency. Applying $s=j\omega$ we get frequency characteristic $H(\omega)=H(s=j\omega)$ of the filter. It is the complex function in frequency domain, which might be represented as magnitude and phase characteristics.

2.1. Fundamentals of filter description

Transfer function of analog filter is represented by the rational function $H(s)$ assumed usually in the form

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \quad (2.1)$$

Assuming $s=j\omega$, we get $H(\omega)=H(s=j\omega)$, which might be represented in the polar form[4,16]

$$H(j\omega) = |H(\omega)| e^{j\phi(\omega)} \quad (2.2)$$

in which $|H(\omega)|$ is the **magnitude characteristic** and $\phi(\omega)$ **phase characteristic**. Magnitude characteristic is of even and phase of odd character with respect to frequency. It means $|H(\omega)| = |H(-\omega)|$ and $\phi(\omega)=-\phi(-\omega)$. Magnitude characteristic is usually expressed in a logarithmic form $20\log_{10}(\text{abs}(H(j\omega)))$ using decibel dB units. Together with the phase characteristic they are called **Bode plot** [7].

In classical filtering process we define so called ideal filters, which are able to stop or pass the signals in certain ranges of frequency. Ideal filter assumes $|H(\omega)| = 1$ in passband and $|H(\omega)| = 0$ in stopband. In classical filtration four types of filters are typically recognized.

- Low-pass filter
- Band-pass filter
- High-pass filter
- Band-stop filter

From the point of view of phase characteristic we can recognize also the filters of linear character (constant group delay) for which output signal $y(t)$ is only scaled and delayed version of the input signal $u(t)$. It means $y(t)=Au(t-t_0)$. The other type of filters are the allpass filters, which have constant magnitude characteristic equal 1 and phase characteristic which is function of frequency $\phi(\omega)$.

In this chapter we will concentrate only on filters from the magnitude point of view. Fig. 2.1 presents the magnitude ideal characteristics of the above mentioned four types of filters [14,16].

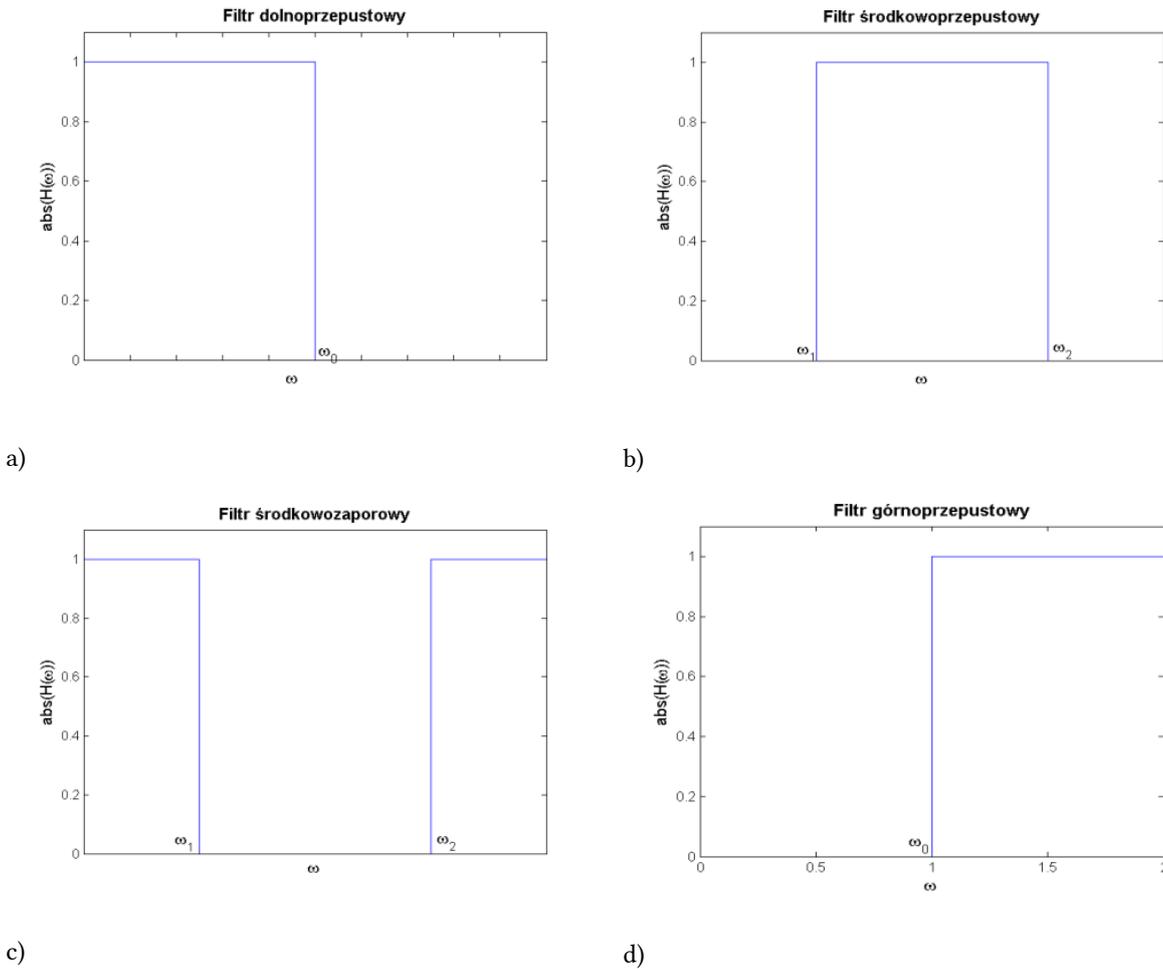


Fig. 2.1. Magnitude characteristics of 4 types of filters: a) low-pass, b) band-pass, c) band-stop, d) high-pass.

In the case of low-pass and high-pass filters the single cutoff frequency f_0 (often represented by angular frequency ω_0) is defined. In the case of band-pass and band-stop filters two cut-off frequencies are recognized: the lower ω_1 and higher ω_2 , as shown in Fig. 2.1.

Practical implementation of ideal characteristics is impossible, because it would require infinite number of elements. Therefore, the approximation of ideal characteristics is used in practice. In such approach instead of gain 1 in passband the value smaller than 1 (assumed by the user) is accepted. In a similar way the value of zero gain in stopband is substituted by the higher one, still accepted from practical point of view (for example 10^{-3}). Moreover, the transit from passband to stopband must be smooth (not of jump character). It means introduction of the third band, called transit band.

2.2. Approximation of frequency characteristics

The approximation formulas should allow controlling the shape of characteristics in passband and stopband. The most important is to preserve the edges of characteristics in the boundary of pass and stop bands. To simplify the approximation process the design formulas are built only for low-pass filters. All other types of filters (band-pass, band-stop and high-pass) follow from the simple transformations defined on the basis of the designed low-pass filter.

In practical approximation procedure concerning the low-pass filters four parameters should be assumed first:

- Frequency f_p (angular frequency ω_p) of the highest value of passband frequency. It corresponds to the frequency at which the maximum attenuation R_p is met or, in other words, the gain achieves the minimum required value G_p . Note that $G_p=1/R_p$ in linear scale or $G_p=-R_p$ in logarithmic scale.
- Frequency f_s (angular frequency ω_s) of the lowest value of stopband frequency. It corresponds to the frequency at which the minimum attenuation R_s is met, or in other words, the gain achieves the maximum allowed value G_s . Similarly as before $G_s=1/R_s$ in linear scale or $G_s=-R_s$ in logarithmic scale.
- Maximum attenuation R_p (minimum gain G_p) in passband, expressed usually in logarithmic scale in

$$R_p = -20 \log_{10} \frac{|U_2|}{|U_1|} \quad [\text{dB}].$$

decibels (

- Minimum attenuation R_s (maximum gain G_s) in stopband, expressed usually in decibels.

Fig. 2.2 illustrates the interpretation of the defined above parameters in linear scale (the upper subplot) and in logarithmic scale in dB (the bottom subplot) for the low-pass filters. The shape of filter characteristic in passband (the range $[G_p, 1]$) or in stopband (the range $[0, G_s]$) does not matter in practice. It might be maximally flat or contain some oscillations (ripples) within the specified ranges.

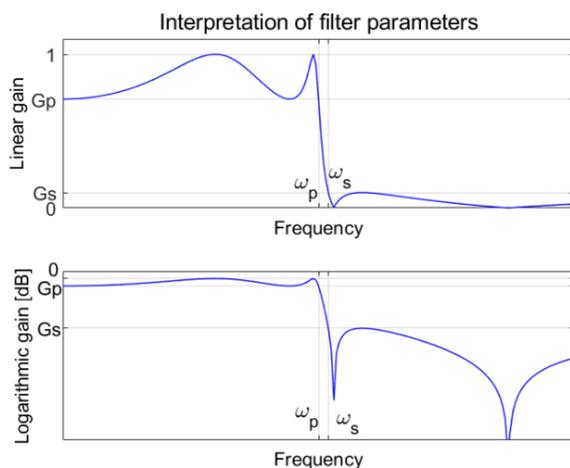


Fig. 2.2. Graphical interpretation of the parameters of low-pass filter, which are the basis in approximation procedure: a) linear scale, b) logarithmic scale.

The approximation procedure should result in a stable system, which means that all poles should be placed in left half-plane of the complex frequency $s=\sigma+j\omega$. Moreover, the results should be presented in the form of rational transfer function $H(s)$. Three different approaches to approximation are typically applied in practice

- Butterworth approximation leading to **maximally flat** magnitude characteristic.
- Chebyshev approximation (**equiripple magnitude** characteristics in the passband or the stopband).
- Elliptic approximation (**equiripple magnitude** characteristics in both the pass and stop bands).

2.2.1. Butterworth approximation

Butterworth filters assume in the initial step the following approximation formula [14,16]

$$|H(\omega)| = \frac{1}{\sqrt{1 + \varepsilon^2 (\omega/\omega_0)^{2n}}} \quad (2.3)$$

In which n is the future order of filter, ε is the ripple factor defining the maximum allowed attenuation in passband ($\varepsilon=1$ corresponds to the attenuation equal 3dB for the angular frequency $\omega=\omega_0$), and ω_0 is the normalized cut-off frequency of the filter (more often in this stage we assume $\omega_0=1$).

Observe that $\varepsilon=1$ leads to $|H(0)| = 1$, $|H(\infty)| = 0$, $|H(\omega = \omega_0)| = 1/\sqrt{2}$ ($R_p=3\text{dB}$ in logarithmic scale). The only parameter in such case ($\varepsilon=1$) is the order n of the filter. It should satisfy the condition

$$n \geq \frac{\lg\left(\frac{10^{0,1R_p}-1}{10^{0,1R_s}-1}\right)}{\lg\left(\frac{\omega_p}{\omega_s}\right)} \quad (2.4)$$

The process of approximation is simplified to transform the proposed form of $|H(\omega)|$ defined by eq. (2.3) into the rational form $H(s)$. It is done by using the following relation

$$|H(\omega)|^2 = H(s)H(-s)|_{s=j\omega} = \frac{1}{1 + (\omega/\omega_0)^{2n}} = \frac{1}{1 + (-1)^n (s/\omega_0)^{2n}} \quad (2.5)$$

The poles s_k of this function are described by the following formula

$$1 + (-1)^n (s/\omega_0)^{2n} = 0 \rightarrow s_k = \omega_0 e^{j\pi(1-n+2k)/2n} \quad (2.6)$$

At $\omega_0=1$ the poles are placed on the unity circle symmetrically in the left and right half-plane. The poles placed in left half-plane are associated with the transfer function $H(s)$

$$s_k = \omega_0 e^{j\pi(1+2k)/2n} e^{-j\pi/2} \quad (2.7)$$

for $k=0, 1, 2, \dots, n-1$ and poles in right half-plane with $H(-s)$. Only $H(s)$ is of interest for us.

As a result the identified transfer function is defined in the form

$$H(s) = (-1)^n \prod_{k=0}^{n-1} \frac{s_k}{s - s_k} \quad (2.8)$$

At $\omega_0=1$ this function assumes the form $H(s)=1/M(s)$, where $M(s)$ is the polynomial of n th order. Matlab uses two functions to approximate Butterworth filter. They are called *buttap* or *butter* with the option 's' [12]. Table 2.1 shows the formulas of Butterworth transfer function $H(s)$ of $\varepsilon=1$ for $n=1, 2, 3, 4$, generated by *buttap* of Matlab.

Table 2.1. Transfer function of Butterworth approximation of different order and $R_p=3\text{dB}$.

$$\begin{aligned}
 n & H(s) \\
 1 & \frac{1}{s+1} \\
 2 & \frac{1}{s^2 + 1.41s + 1} \\
 3 & \frac{1}{(s+1)(s^2 + s + 1)} \\
 4 & \frac{1}{(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)}
 \end{aligned}$$

Fig. 2.3. depicts the magnitude characteristics of Butterworth filter for different orders: $n=2, 6, 10$ and $R_p=3\text{dB}$. The characteristic is maximally flat. The higher the order the more steep is transfer form passband to stopband range.

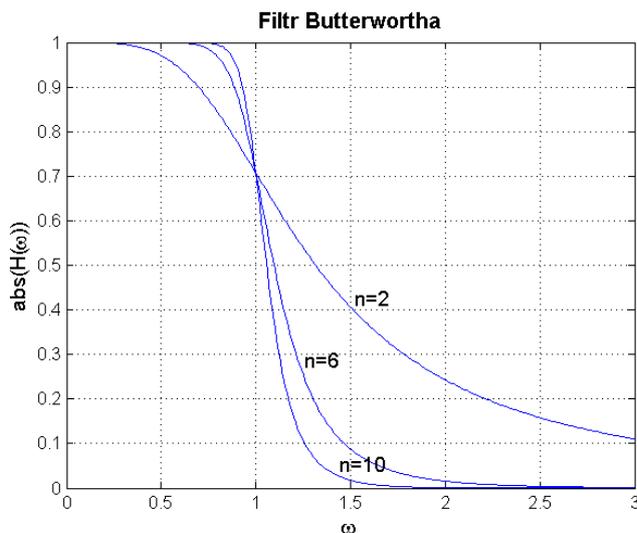


Fig. 2.3. Magnitude characteristics of Butterworth filters for $n=2, 6, 10$ and $R_p=3\text{dB}$.

2.2.2. Chebyshev approximation

In the case of Chebyshev approximation there are two solutions: type I and type II (the latter called also inverse form). In the first case the initial form of approximation of magnitude is given as follows [14,16]

$$|H(\omega)| = \frac{1}{\sqrt{1 + \varepsilon^2 T_n^2(\omega/\omega_0)}} \quad (2.9)$$

where $T_n(x)$ means **Chebyshev polynomial** of the first kind and n th order. The parameter ω_p is the cutoff

angular frequency of low-pass filter, and ϵ the ripple factor characterizing allowed attenuation in passband. The Chebyshev polynomials of the first kind are defined by the [recurrence relation](#)

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad (2.10)$$

for $n=2, 3, 4, \dots$ at $T_0(x)=1, T_1(x)=x$. Table 2.2 presents the forms of Chebyshev polynomials of the first kind and 4 different orders $n=1, 2, 3, 4$.

Table 2.2. Chebyshev polynomials of the first kind.

n	$T_n(x)$
1	x
2	$2x^2-1$
3	$4x^3-3x$
4	$8x^4-8x^2+1$

It is evident now, that magnitude characteristic of $H(s)$ defined by (2.9) satisfy conditions $|H(0)| = 1$ for even values of n and $|H(0)| = 1/\sqrt{1 + \epsilon^2}$ for odd n . Similarly we have $|H(1)| = 1/\sqrt{1 + \epsilon^2}$ for all values of n . The values of ripple factor ϵ and the order n of the required filter are calculated as follows

$$\epsilon = \sqrt{10^{0,1R_p} - 1} \quad (2.11)$$

$$n \geq \frac{\cosh^{-1} \sqrt{\frac{10^{0,1R_s} - 1}{\epsilon^2}}}{\cosh^{-1} \frac{\omega_s}{\omega_p}} \quad (2.12)$$

Further procedure transforming $|H(\omega)|$ into $H(s)$ is similar to the Butterworth procedure. However, this time the poles are located on the ellipse (not the circle as in Butterworth case). Matlab uses two functions to approximate Chebyshev filter of type I. They are called *cheb1ap* or *cheby1* with the option 's' [12]. Table 2.3 shows four expressions for Chebyshev transfer function $H(s)$ for $n=1, 2, 3, 4$.

Table 2.3. Transfer functions of Chebyshev filter for $\epsilon=1$ ($R_p=3\text{dB}$) and $n=1, 2, 3, 4$.

n H(s)

$$1 \quad \frac{1}{s + 1}$$

$$2 \quad \frac{0,5}{s^2 + 0.644s + 0.707}$$

$$3 \quad \frac{0.25}{s^3 + 0.596s^2 + 0.928s + 0.25}$$

$$4 \quad \frac{0,125}{s^4 + 0,581s^3 + 1,169s^2 + 0,404s + 0,177}$$

Fig. 2.4 presents the magnitude characteristics of Chebyshev filter of kind I for chosen values of $n=2, 6, 10$ and $R_p=1\text{dB}$. Equal ripples in the passband and flat characteristic in stopband can be observed. The number of ripples depends on the order of filter. The higher the order n the steeper is transfer from passband to stopband range. For any value of n the Chebyshev characteristic is more steep than Butterworth one.

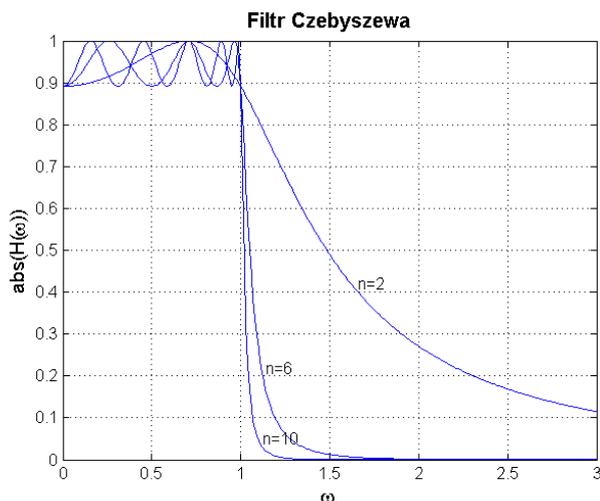


Fig. 2.4. Magnitude characteristics of Chebyshev filters of the type I for $n=2, 6, 10$ and value $R_p=1\text{dB}$.

The second type of Chebyshev filter (inverse filter) is also applied in practice. It starts from the approximation formula given as follows [14,16]

$$|H(\omega)| = \frac{\varepsilon^2 T_n^2(\omega_0/\omega)}{\sqrt{1 + \varepsilon^2 T_n^2(\omega_0/\omega)}} \quad (2.13)$$

The parameters have the same sense as in type I filter. The poles p_{ki} of the inverse filter are strictly associated with the poles p_k of type I filter and are related in the form

$$p_{ki} = -\frac{1}{p_k} \quad (2.14)$$

There are two Matlab functions used in designing type II Chebyshev filter: *cheb2ap* or *cheby2* with the option 's'. The magnitude frequency characteristic of inverse filter is maximally flat in passband and of equal ripples in the stopband.

2.2.3. Elliptic approximation

The starting point in elliptic approximation is assumed in the form [14,16]

$$|H(\omega)| = \frac{1}{\sqrt{1 + \varepsilon^2 R_n^2(\omega/\omega_0)}} \quad (2.15)$$

where ω_0 is the cut off angular frequency (usually $\omega_0=1$), ε the ripple factor and $R_n(\omega/\omega_0)$ represents the rational function of n th order. In this approximation the $R_n(\omega)$ is described as follows:

- for odd value of n

$$R_n(\omega) = \omega \prod_{i=1}^{(n-1)/2} \frac{\omega_i^2 - \omega^2}{1 - \omega_i^2 \omega^2} \quad (2.16)$$

- for even value of n

$$R_n(\omega) = \prod_{i=1}^{n/2} \frac{\omega_i^2 - \omega^2}{1 - \omega_i^2 \omega^2} \quad (2.17)$$

On the basis of design specifications given by four parameters ($\omega_p, \omega_s, R_p, R_s$) the order n and ripple factor ε should be calculated. The ε value is calculated from the same expression as in the previous approach

$$\varepsilon = \sqrt{10^{0,1R_p} - 1} \quad (2.18)$$

The order n of the filter requires more steps. Assuming $k=\omega_p/\omega_s$ the following steps should be done [19]

$$q_0 = 0,5 \frac{1 - (1 - k^2)^{1/4}}{1 + (1 - k^2)^{1/4}}$$

$$q = q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13}$$

$$d = \sqrt{\frac{10^{0,1R_p} - 1}{10^{0,1R_s} - 1}} \quad \text{latex (2.19)}$$

Finally the order n can be determined as follows

$$n \geq \frac{\lg(16/d^2)}{\lg(1/q)} \quad (2.20)$$

The procedure of transferring $|H(\omega)|$ to $H(s)$ is more complicated. The details can be found in specialized books, for example handbooks devoted to filter design [14,19]. Matlab has developed two functions that might be used in generating elliptic approximation: *ellipap* or *ellip* with option 's'. Table 2.4 presents exemplary forms of $H(s)$ for different orders of filter $n=1, 2, 3, 4$ and $R_p=3\text{dB}$ ($\varepsilon=1$) and $R_s=40\text{dB}$.

Table 2.4. Transfer functions of elliptic filters for $\varepsilon=1$ ($R_p=3\text{dB}$) and $R_s=40\text{dB}$

n H(s)

$$1 \quad \frac{1.0024}{s + 1.0024}$$

$$2 \quad \frac{0.01s^2 + 0.5062}{s^2 + 0.6403s + 0.7150}$$

$$3 \quad \frac{0.0553s^2 + 0.2790}{s^3 + 0.5910s^2 + 0.9499s + 0.2790}$$

$$4 \quad \frac{0.01s^4 + 0.1101s^2 + 0.1815}{s^4 + 0.5736s^3 + 1.2654s^2 + 0.4589s + 0.2564}$$

Fig. 2.5 presents the magnitude characteristics of elliptic filters for three different orders: $n=2, 6, 10$, and $\omega_0=1$, $R_p=1\text{dB}$ and $R_s=40\text{dB}$. Equal ripples in both: passband and stopband are observed. The number of oscillations depends on the value of n . For the values of $n>2$ we observe very steep transfer from passband to stopband. At $n=10$ the filter resembles the ideal characteristic.

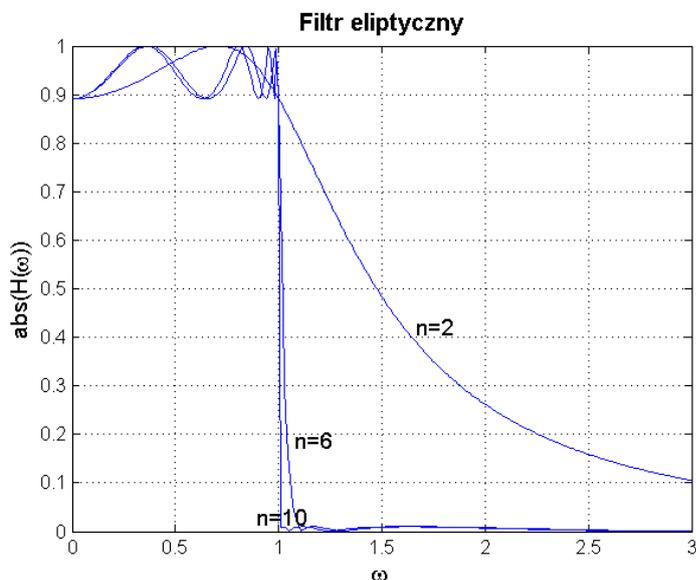


Fig. 2.5. Magnitude characteristics of elliptic filters for $n=2, 6, 10$ and $R_p=1\text{dB}$, $R_s=40\text{dB}$.

Comparing the characteristics of three presented filters it is evident that for the same order the best (closest to ideal) is the elliptic filter. The least ideal is the Butterworth approximation. This is well seen on the example of 6th order filters (Fig. 2.6) of the same specifications ($n=6$, $\omega_0=0.5$, $R_p=1\text{dB}$, $R_s=40\text{dB}$) but applying different approximations.

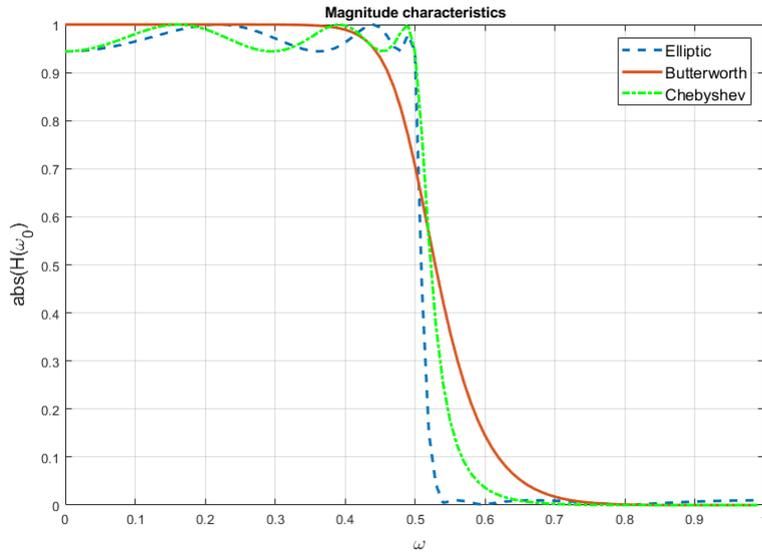


Fig. 2.6. Comparison of magnitude characteristics of filters according to different approximations. Filter specifications: $n=6$, $\omega_0=0.5$, $R_p=1\text{dB}$, $R_s=40\text{dB}$.

Table 2.5 presents the required values of order n of these three filters at constant values of $R_p=3\text{dB}$ i $R_s=40\text{dB}$ and changing values of ω_p and ω_s . The order of filters depend on the ratio ω_s/ω_p which were assumed as 2, 1,5 and 1,05 (the steepest transfer from passband to stopband).

Table 2.5. Comparison of required order of Butterworth, Chebyshev and elliptic filters for the same specifications: ω_p , ω_s , $R_p=3\text{dB}$ i $R_s=40\text{dB}$.

	Butterworth filter	Chebyshev filter	Elliptic filter
$\omega_s/\omega_p=2$	$n=13$	$n=8$	$n=3$
$\omega_s/\omega_p=1,5$	$n=23$	$n=11$	$n=4$
$\omega_s/\omega_p=1.05$	$n=189$	$n=32$	$n=8$

The advantage of the elliptic approximation is visible irrespective of the required values of design specifications.

2.3. Frequency transformations

The presented above design procedures have been defined for only low-pass, normalized filter of $\omega_0=1$. Transformation of this designed filter to the proper type and real values of frequency parameters needs only small amount of computations, which will be presented in the next sections.

2.3.1. Transformation of normalized low-pass filter to filter of real parameters

The normalized low-pass filter of $\omega_0=1$ will be transformed to the low-pass filter of the real required frequency specifications ω_{0r} using the following substitution

$$s \rightarrow \frac{s}{\omega_{0r}} \quad (2.21)$$

The unity angular frequency $\omega_0=1$ will be transformed to the real one ω_{0r} without the change of the filter order.

Example 2.1

Let us illustrate this procedure on the example of Butterworth filter of the second order

$$H(s) = \frac{1}{s^2 + 1.41s + 1}$$

Applying $\omega_{0r}=1000$ we get the final form of transfer function

$$H(s) = \frac{10^6}{s^2 + 1410s + 10^6}$$

Fig. 2.7 shows the comparison of both filters: the normalized (upper) and the real one (bottom).

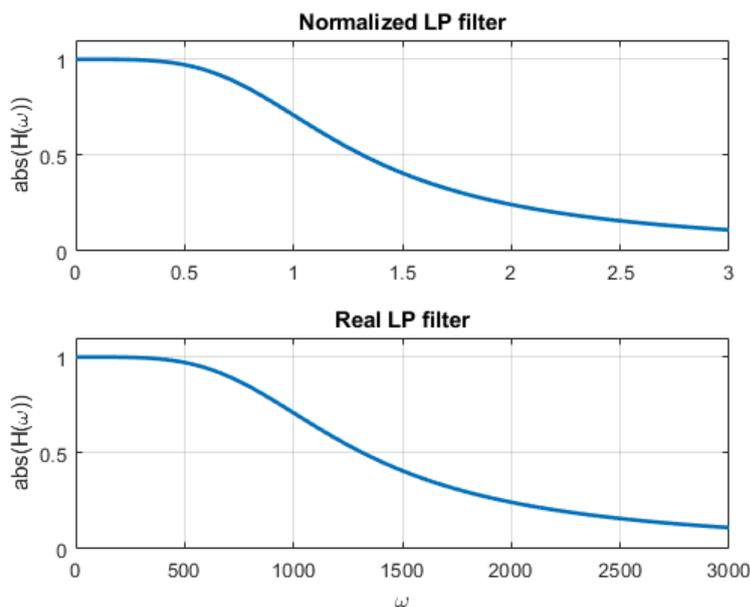


Fig. 2.7. Magnitude characteristics of the low-pass Butterworth filter of second order. The upper plot presents the normalized and bottom – the real characteristic after transformation.

Analog filters are implemented using RLC elements arranged in specific structure. Usually the values of elements are calculated for normalized transfer function. Let us assume that their computed values are equal R_n , L_n and C_n . Frequency transformation does not change the value of resistors, however, does transform the capacitance and inductance values. Changing s to s/ω_{0r} transforms L and C to the real values according to the following relations

$$Z_{L_n} = sL_n \rightarrow Z = s \frac{L_n}{\omega_{0r}}$$

$$Y_{C_n} = sC_n \rightarrow Y = s \frac{C_n}{\omega_{0r}} \quad (2.22)$$

It means that L_n is changed to $L=L_n/\omega_{0r}$, and C_n to $C=C_n/\omega_{0r}$, leaving the resistance values unchanged (they don't depend on frequency).

2.3.2. Transformation of normalized low-pass to real high-pass filter

The normalized low-pass filter of $\omega_0=1$ is transformed to high-pass type of real ω_{0r} by applying the following substitution [14,16]

$$s \rightarrow \frac{\omega_{0r}}{s} \quad (2.23)$$

This transformation changes not only the normalized frequency $\omega_0=1$ into the required one ω_{0r} but transforms also the type of the filter to high-pass. The order of the transfer function remained the same.

Example 2.2

To illustrate the procedure we will start from the same 2nd order Butterworth filter of the normalized transfer function

$$H(s) = \frac{1}{s^2 + 1.41s + 1}$$

Let us change it to the high-pass real filter of $\omega_{0r}=1000$. Application of eq. (2.23) leads to the following transfer function of high-pass characteristic

$$H(s) = \frac{s^2}{s^2 + 1410s + 10^6}$$

Fig. 2.8 presents both magnitude characteristics: the normalized low-pass and the transformed high-pass.

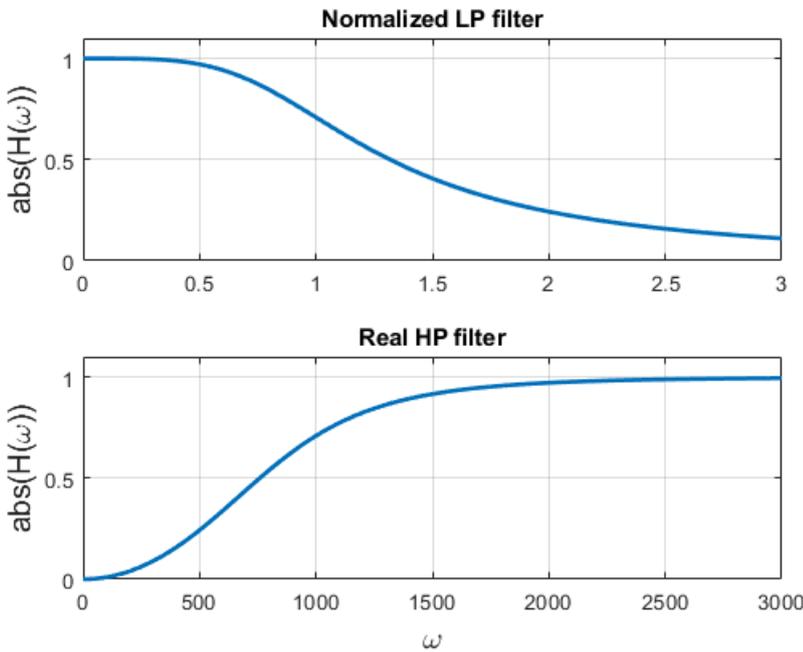


Fig. 2.8. Magnitude characteristics of the filters: the normalized low-pass prototype (upper) and the transformed high-pass (bottom).

Transformation of frequency changes also the character and values of L and C elements in analog implementation of the filter according to the expressions

$$Z_{L_n} = sL_n \rightarrow Z = \omega_{0r}L_n \frac{1}{s}$$

$$Y_{C_n} = sC_n \rightarrow Y = \omega_{0r}C_n \frac{1}{s} \quad (2.24)$$

It means that normalized L_n is converted to the capacitance $C=1/(\omega_{0r}L_n)$, and normalized C_n into inductance $L=1/(\omega_{0r}C_n)$. The resistor was not changed after the transformation.

2.3.3. Transformation of normalized low-pass to real band-pass filter

Transformation of the normalized low-pass filter into real band-pass one of the boundary frequencies ω_1 and ω_2 needs introducing two new parameters: the width B of the pass band and the center frequency ω_0 . They are defined as follows [14,16]

$$\begin{aligned} B &= \omega_2 - \omega_1 \\ \omega_0 &= \sqrt{\omega_1 \omega_2} \end{aligned} \quad (2.25)$$

In such case the normalized low-pass filter is converted to band-pass filter of real values of parameters using the following substitution

$$s \rightarrow \frac{s^2 + \omega_0^2}{Bs} \quad (2.26)$$

Observe that in this case the order of the resulting band-pass filter has been doubled with respect to the normalized prototype.

Example 2.3

To illustrate the procedure we will start from the same 2nd order Butterworth filter of the normalized transfer function

$$H(s) = \frac{1}{s^2 + 1.41s + 1}$$

Assume $\omega_1=100$ and $\omega_2=200$. At such values $B=100$ and $\omega_0= 141.42$. Application of eq. (2.26) leads to the following solution

$$H(s) = \frac{10^4 s^2}{s^4 + 141s^3 + 5 \cdot 10^4 s^2 + 2.82 \cdot 10^6 s + 4 \cdot 10^8}$$

Fig. 2.9 presents both magnitude characteristics: the normalized low-pass and the transformed band-pass.

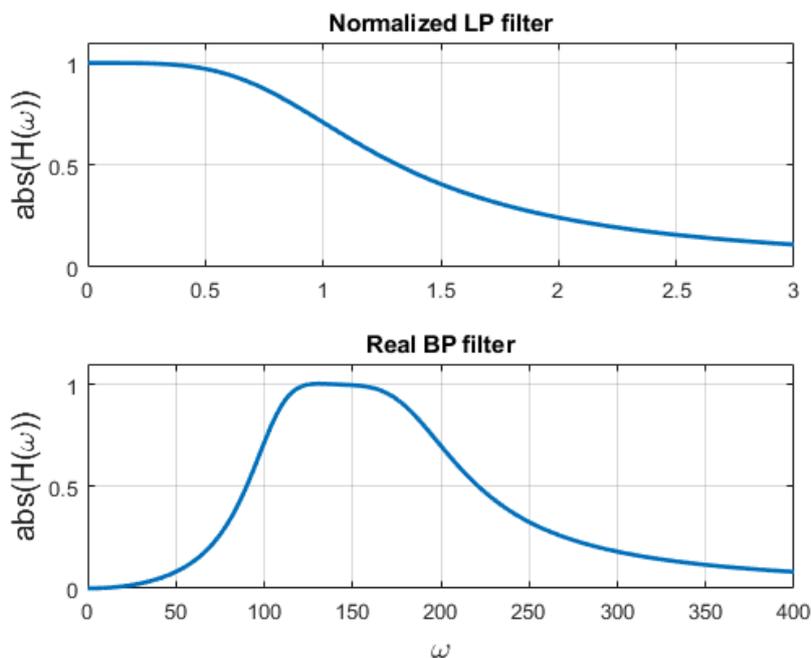


Fig. 2.9. Magnitude characteristics of the filters: the normalized low-pass prototype (upper) and the transformed band-pass (bottom).

Transformation of frequency changes also the character and values of L and C elements in analog implementation of the filter according to the expressions

$$\begin{aligned} Z_{L_n} = sL_n \rightarrow Z &= L_n \frac{s^2 + \omega_0^2}{Bs} = \frac{L_n}{B} s + \frac{L_n \omega_0^2}{B} \frac{1}{s} \\ Y_{C_n} = sC_n \rightarrow Y &= C_n \frac{s^2 + \omega_0^2}{Bs} = \frac{C_n}{B} s + \frac{C_n \omega_0^2}{B} \frac{1}{s} \end{aligned} \quad (2.27)$$

It means, that in analog implementation of such filter at application of RLC elements, the normalized

inductance L_n is converted to the series connection of $C = \frac{B}{\omega_0^2 L_n}$ and inductance $L = \frac{L_n}{B}$. On the other side the normalized capacitance C_n of the prototype filter is converted to the parallel connection of inductance

$$L = \frac{B}{\omega_0^2 C_n} \quad \text{and capacitance} \quad C = \frac{C_n}{B}. \quad \text{The resistors have been left without changes.}$$

2.3.4. Transformation of normalized low-pass to real band-stop filter

Transformation of the normalized low-pass filter into real band-stop one of the boundary frequencies ω_1 and ω_2 needs, similarly as in the case of previous transformation, introducing two new parameters: the width B of the stop band and the center frequency ω_0 [14,16]

$$\begin{aligned} B &= \omega_2 - \omega_1 \\ \omega_0 &= \sqrt{\omega_1 \omega_2} \end{aligned} \quad (2.28)$$

In such case the normalized low-pass filter is converted to band-stop filter of real values of parameters using the substitution

$$s \rightarrow \frac{Bs}{s^2 + \omega_0^2} \quad (2.29)$$

In this case the order of the resulting band-stop filter has been doubled with respect to the normalized prototype.

Example 2.4

As an example consider the 2nd order normalized Butterworth filter described by

$$H(s) = \frac{1}{s^2 + 1.41s + 1}$$

Assume in transformation $\omega_1=100$ and $\omega_2=200$. It means $B=100$ and $\omega_0= 141.42$. After applying (2.29) we get the final form of band-stop filter

$$H(s) = \frac{s^4 + 4 \cdot 10^4 s^2 + 4 \cdot 10^8}{s^4 + 141s^3 + 5 \cdot 10^4 s^2 + 2.82 \cdot 10^6 s + 4 \cdot 10^8}$$

Fig. 2.10 presents both magnitude characteristics: the normalized low-pass and the transformed band-stop.

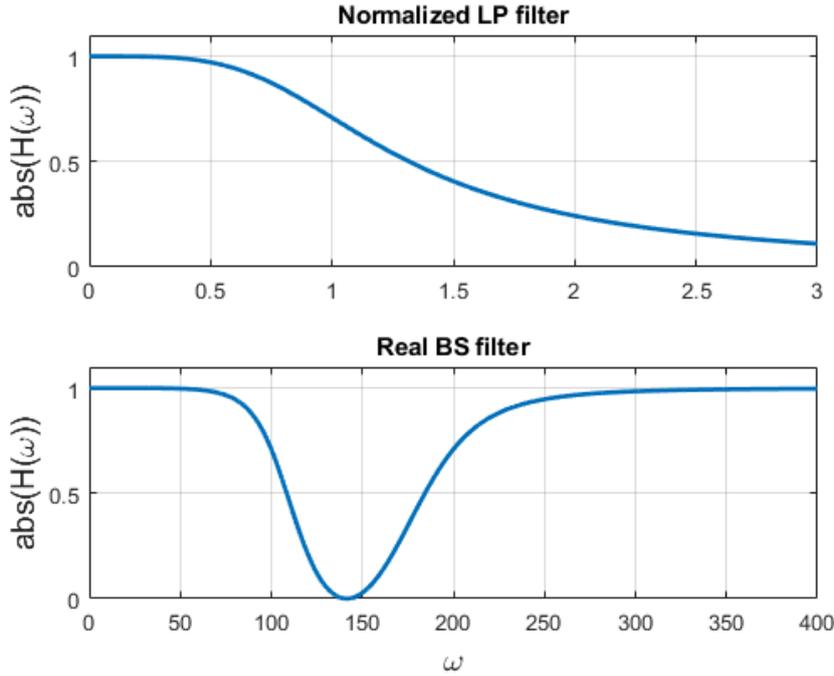


Fig. 2.10. Magnitude characteristics of the filters: the normalized low-pass prototype (upper) and the transformed band-stop (bottom).

Transformation of frequency changes also the character and values of L and C elements in analog implementation of the filter according to the expressions

$$\begin{aligned}
 Y_{L_n} = \frac{1}{sL_n} &\rightarrow Y = \frac{s^2 + \omega_0^2}{BL_n s} = \frac{1}{BL_n} s + \frac{\omega_0^2}{BL_n} \cdot \frac{1}{s} \\
 Z_{C_n} = \frac{1}{sC_n} &\rightarrow Z = \frac{s^2 + \omega_0^2}{BC_n s} = \frac{1}{BC_n} s + \frac{\omega_0^2}{BC_n} \cdot \frac{1}{s}
 \end{aligned} \quad (2.30)$$

It means that in analog implementation of such filter at application of RLC elements the normalized

inductance L_n is converted to the parallel connection of $C = \frac{1}{BL_n}$ and inductance $L = \frac{BL_n}{\omega_0^2}$. On the other side the normalized capacitance C_n of the prototype filter is converted to the series connection of

inductance $L = \frac{1}{BC_n}$ and capacitance $C = \frac{BC_n}{\omega_0^2}$. The resistors are left without changes.

The presented frequency transformations did not change the values of resistors. In practical calculations of element using the normalized transfer function the values of resistors are usually close to one, not practical from implementation point of view. Therefore, the resistance scaling is necessary. Observe, that voltage transfer function is defined as the ratio of impedances. Therefore the application of resistance scaling does not change the coefficients of transfer function. However, this time all impedances representing also capacitors and inductors should be scaled in the same way.

Assume all element values after design procedure were normalized and equal L_n, C_n, R_n , respectively. Their corresponding values in real life will be denoted with index r . Assume resistance scaling ratio equal k_r and ratio of frequency scaling equal k_f . The scaling ratio is defined in the following way

$$k_r = \frac{R_r}{R_n}$$

$$k_f = \frac{f_r}{f_n} \quad (2.31)$$

Scaling the normalized values to the real ones will be done using the following expressions

$$R_r = k_r R_n$$

$$X_{Lr} = \omega_r L_r = k_R \omega_n L_n \rightarrow L_r = L_n k_R \frac{\omega_n}{\omega_r} = L_n \frac{k_R}{k_f}$$

$$X_{Cr} = \frac{1}{\omega_r C_r} = \frac{k_R}{\omega_n C_n} \rightarrow C_r = C_n \frac{1}{k_R} \frac{\omega_n}{\omega_r} = C_n \frac{1}{k_R k_f} \quad (2.32)$$

Example 2.5

Let us assume normalized values of elements given as follows: $R_{1n}=0.5\Omega$, $R_{2n}=0.7\Omega$, $C_{1n}=2F$, $C_{2n}=1.5F$, $L=1H$ (not practical values in implementation). Assume scaling coefficients as following: $k_r=10^4$ (resistance equal one will be changed to 10k Ω) and $k_f=10^5$ (unity frequency changed to 10^5). Applying these values to eq. (2.32) we get their real counterparts as follows

$$R_{1r}=5k\Omega,$$

$$R_{2r}=7k\Omega,$$

$$C_{1n}=2nF,$$

$$C_{2n}=1.5nF,$$

$$L=0.1H$$

2.4. Higher order filters in cascade connection of biquadratic blocks

As a result of approximation procedure we obtain the transfer function $H(s)=L(s)/M(s)$ which represents stable system (the poles in left half-plane). However, the order of polynomial of denominator is usually high. In practical implementation of the coefficients of this polynomial the values of resistances, capacitances and inductances are never ideally equal to the computed ones. Therefore, the poles of the system might move to the right half-plane of complex frequency, which means unstable operation of the system.

Observe that not ideal setting the values of elements will change the coefficients of the denominator $M(s)$ and also the pole locations

$$M(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = \prod_{i=1}^n (s - p_i) \quad (2.33)$$

Because of

$$a_k \rightarrow a_k + \Delta a_k \quad (2.34)$$

we get $M'(s)$ instead of $M(s)$, where

$$M'(s) = (a_n + \Delta a_n) s^n + (a_{n-1} + \Delta a_{n-1}) s^{n-1} + \dots + (a_1 + \Delta a_1) s + (a_0 + \Delta a_0) \quad (2.35)$$

The poles p_i have been changed by Δp_i according to relation

$$\Delta p_i = \sum_{k=1}^n \frac{\partial p_i}{\partial a_k} \Delta a_k \quad (2.36)$$

The significant is sensitivity of p_i with respect to a_k . We determine it using the relation

$$\frac{\partial M(s)}{\partial a_k} = \frac{\partial M(s)}{\partial p_i} \frac{\partial p_i}{\partial a_k} \rightarrow \frac{\partial p_i}{\partial a_k} = \frac{\frac{\partial M(s)}{\partial a_k}}{\frac{\partial M(s)}{\partial p_i}} \quad (2.37)$$

If we take into account

$$\begin{aligned} \frac{\partial M(s)}{\partial a_k} &= s^k \\ \frac{\partial M(s)}{\partial p_i} &= - \prod_{\substack{l=1 \\ l \neq i}}^n (s - p_l) \end{aligned} \quad (2.38)$$

for $s=p_i$ we get the final form of sensitivity formula

$$\frac{\partial p_i}{\partial a_k} = \frac{-p_i^k}{\prod_{\substack{l=1 \\ l \neq i}}^n (p_i - p_l)} \quad (2.39)$$

Observe that in practical realization of $H(s)$ the differences $(p_i - p_l)$ are very small. The higher the order of the filter the larger is the value of sensitivity. The smallest possible one it achieved for $n=2$ (biquadratic filter). In such case (two poles p_1 and p_2) we get

$$\begin{aligned} \frac{\partial p_1}{\partial a_k} &= \frac{-p_1^k}{(p_1 - p_2)} \\ \frac{\partial p_2}{\partial a_k} &= \frac{-p_2^k}{(p_2 - p_1)} \end{aligned} \quad (2.40)$$

for $k=0, 1$ and 2 . Therefore, to minimize sensitivity of poles to the coefficient values we should decompose the higher order transfer function into second order section (biquadratic) and combine them in cascade. As a result the original transfer function of higher order may be presented as the product of functions of second order.

$$H(s) = \frac{U_{out}}{U_{in}} = H_1(s)H_2(s) \cdots H_m(s) \quad (2.42)$$

This decomposition is done through paring the terms containing the complex conjugate poles.

In similar way the zeros (roots of numerator) are paired and create the second order polynomials of the biquadratic sections. Final form of such decomposition into biquadratic sections connected in cascade is presented in Fig. 2.11.

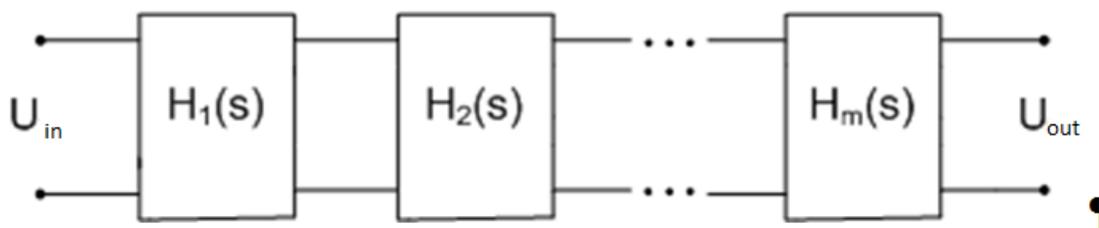


Fig. 2.11. Cascade connection of second order section.

Example 2.6

Let us assume elliptic filter of 6th order of $\omega_0=1$, $R_p=1\text{dB}$, $R_s=40\text{dB}$. The following Matlab lines generate the

polynomials of numerator and denominator of higher order (the first line) and then decompose into second order sections (second line).

```
[L,M]=ellip(6,1,40,1,'s')
[sos,g]=tf2sos(L,M)
```

As a results we get

```
L = 0.0100 0.0000 0.1173 0.0000 0.2800 -0.0000 0.1860 M = 1.0000 0.9154 2.2378 1.4799 1.4317
```

which corresponds to the transfer function of the 6th order

$$H(s) = \frac{0.01s^6 + 0.1173s^4 + 0.2800s^2 + 0.1860}{s^6 + 0.9154s^5 + 2.2378s^4 + 1.4799s^3 + 1.4317s^2 + 0.5652s + 0.2087}$$

The second command generates

```
sos =
1.0000 0.0000 8.7778 1.0000 0.6310 0.2678
1.0000 0.0000 1.7081 1.0000 0.2368 0.7799
1.0000 -0.0000 1.2406 1.0000 0.0476 0.9994
```

```
g =
0.0100
```

which is interpreted as second order sections in the following way

$$H(s) = 0.01 \frac{s^2 + 1.2406}{s^2 + 0.0476s + 0.9994} \cdot \frac{s^2 + 1.7081}{s^2 + 0.2368s + 0.7799} \cdot \frac{s^2 + 8.7778}{s^2 + 0.6310s + 0.2678}$$

Each of these sections is implemented in an independent way. Therefore, the coefficients of one section do not influence the other sections. The computed poles of the biquadratic sections form the following pairs:

```
p1=-0.0237 + j0.9995, p2=-0.0237 - j0.9995,
p3=-0.1185 + j0.8751, p4=-0.1185 - j0.8751,
p5 =-0.3155 + j0.4101, p6 =-0.3155 - j0.4101
```

which are represented in the succeeding biquadratic functions.

2.5. Biquadratic sections

The biquadratic section (called often “biquad”) is presented in general form as follows

$$H_b(s) = \frac{b_2s^2 + b_1s + b_0}{s^2 + a_1s + a_0} = \frac{b_2s^2 + b_1s + b_0}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \tag{2.43}$$

In this form Q means the quality factor and ω_0 – the characteristic center angular frequency. Four specific forms of biquads are usually considered

- Low-pass filter

$$H_{LP}(s) = \frac{b_0}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \frac{A_{LP}\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2} \tag{2.44}$$

- Band-pass filter

$$H_{BP}(s) = \frac{b_1 s}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} = \frac{A_{BP} \frac{\omega_0}{Q} s}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \quad (2.45)$$

- High-pass filter

$$H_{HP}(s) = \frac{b_2 s^2}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} = \frac{A_{HP} s^2}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \quad (2.46)$$

- Band-stop filter

$$H_{BS}(s) = \frac{s^2 + b_0}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} = \frac{s^2 + \omega_z^2}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \quad (2.47)$$

All types of filters have the same form of denominator. They differ only by the polynomial of numerator. The parameters A_{LP} , A_{BP} , A_{HP} represent the gain factor of the particular type of filter. In the case of low-pass filter this value corresponds to $\omega=0$, in the case of band-pass to $\omega=\omega_0$, and in high-pass to $\omega=\infty$. Each of these biquadratic filters has found many practical electronic implementations. We will concentrate in this work to only KHN and the multiloop structure, which were analyzed in the first chapter using SFG.

2.6. Implementation of biquadratic sections using KHN structure

The most often used electronic circuit in biquad implementation is the KHN structure (Fig. 2.12) applying three operational amplifiers.

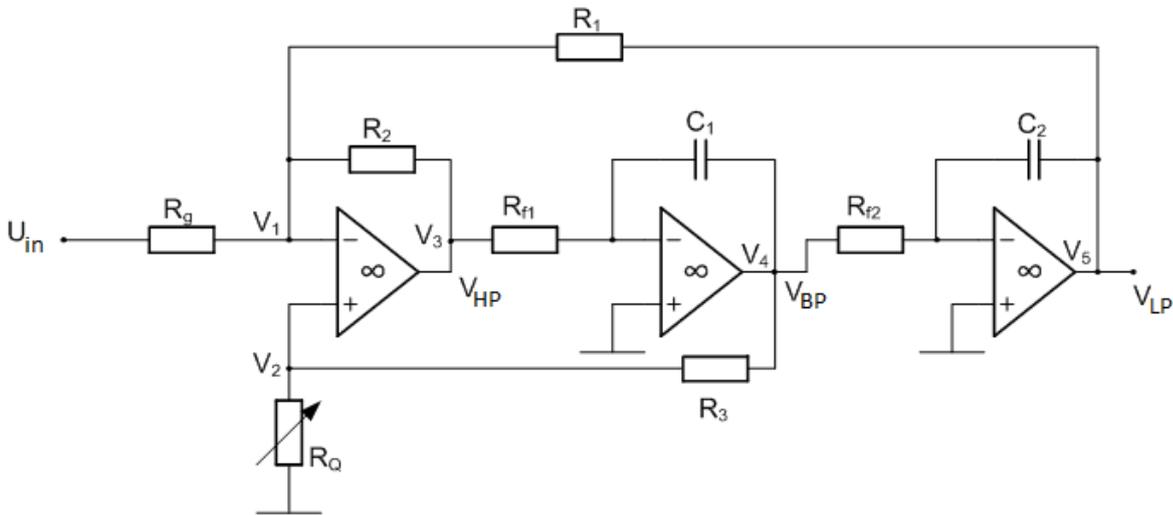


Fig. 2.12. Circuit structure of KHN

The circuit is able to implement three different types of the biquads in the same structure, depending on the assumed output node. They are defined as follows

$$H_{LP}(s) = \frac{V_5}{U_{in}} = \frac{-\frac{R_2}{C_1 C_2 R_{f1} R_{f2} R_g}}{M(s)} \quad (2.47)$$

$$H_{BP}(s) = \frac{V_4}{U_{in}} = \frac{s \frac{R_2}{C_1 R_{f1} R_g}}{M(s)} \quad (2.48)$$

$$H_{HP}(s) = \frac{V_3}{U_{in}} = \frac{-s^2 \frac{R_2}{R_g}}{M(s)} \quad (2.49)$$

The denominator $M(s)$ in all transfer functions is the same and expressed in the following way

$$M(s) = s^2 + s \frac{R_2(G_1 + G_2 + G_g)}{C_1 R_3 R_{f1}(G_3 + G_Q)} + \frac{R_2}{C_1 C_2 R_1 R_{f1} R_{f2}} \quad (2.50)$$

The fourth type of filter (the band-stop) can be implemented as the sum of the low-pass and high-pass filter signals. The important point is that above equations are the starting points in filter design procedure. It can be easily proved that

$$\omega_0 = \sqrt{\frac{R_2}{C_1 C_2 R_1 R_{f1} R_{f2}}} \quad (2.51)$$

$$Q = \left(1 + \frac{R_3}{R_Q}\right) \frac{1}{(1/R_1 + 1/R_2 + 1/R_g)} \sqrt{\frac{R_{f1} C_1}{R_1 R_2 R_{f2} C_2}} \quad (2.52)$$

$$A_{LP} = \frac{R_1}{R_g} \quad (2.53)$$

$$A_{BP} = \left(1 + \frac{R_3}{R_Q}\right) \frac{1}{(1 + R_g/R_1 + R_g/R_2)} \quad (2.54)$$

$$A_{HP} = \frac{R_2}{R_g} \quad (2.55)$$

In the design procedure of second order filters we want to obtain the required value of Q and characteristic center frequency ω_0 . The gains of the particular filter types are defined by equations (2.53-2.55). The design is aimed in calculating the values of elements providing the required values of Q and frequency ω_0 . Usually the following simplifying constraints are assumed

$$C_1 = C_2 = C$$

$$R_{f1} = R_{f2} = R_f$$

$$R_1 = R_2 = R_3 = R \quad (2.56)$$

These parameters can take any values, which are required from technological point of view. At such assumption we get finally

$$R_f = \frac{1}{\omega_0 C} = \frac{1}{2\pi f_0 C}$$

$$R_g = \frac{R}{A_{DP}} \text{ lub } R_g = \frac{R}{A_{GP}} \text{ lub } R_g = \frac{RQ}{A_{SP}}$$

$$R_Q = \frac{R}{2Q + QR/R_g - 1}$$

The values of R and C may be arbitrary, for example $R=10\text{k}\Omega$, $C=1\text{nF}$. In this way we obtain immediately the

real values of parameters of the filters, without the need to de-normalize them. The main advantage of KHN structure is possibility of realizing three types of filters in the same structure (changing only the output node). Moreover, the parameters of biquad (cutoff frequency and quality factor) are fitted in an independent way using different elements of the circuit. Changing the value of Q does not influence the cutoff frequency.

2.7. Example of biquadratic filter in multiloop structure

The other prototype used in filter design is the multiloop circuit structure applying only one ideal operational amplifier which is presented in Fig. 2.13.

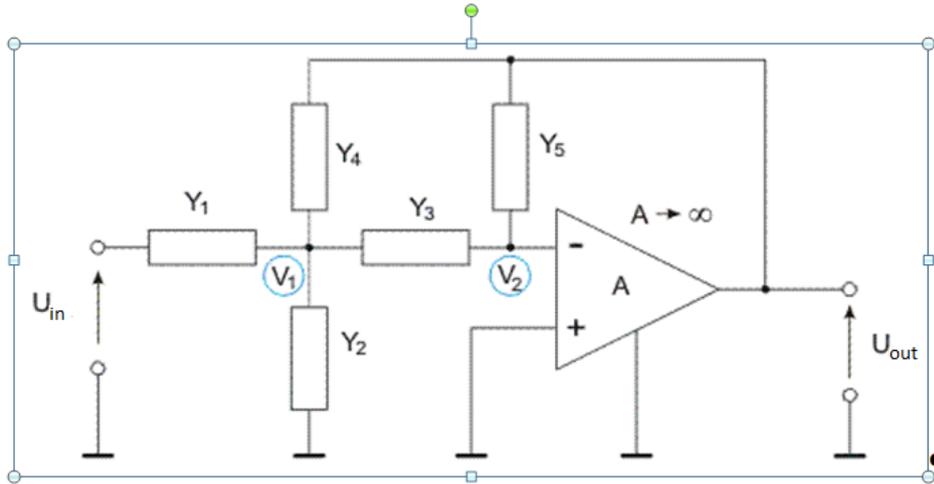


Fig. 2.13 Multiloop circuit structure as the prototype for biquadratic filters

Its transfer function obtained in chapter 1 is of the form

$$H(s) = \frac{U_{out}}{U_{in}} = \frac{-Y_1 Y_3}{Y_5 (Y_1 + Y_2 + Y_3 + Y_4) + Y_3 Y_4} \quad (2.48)$$

By using different types of elements we can adopt this structure to realize three types of biquadratic filters.

Low-pass filter

It can be realized in the multiloop structure by adjusting the following types of elements

$$Y_1 = G_1, Y_2 = sC_2, Y_3 = G_3, Y_4 = G_4, Y_5 = sC_5,$$

In such case the transfer function is of the form

$$H_{LP}(s) = \frac{-\frac{G_1 G_3}{C_2 C_5}}{s^2 + \frac{G_1 + G_3 + G_4}{C_2} + \frac{G_3 G_4}{C_2 C_5}} \quad (2.49)$$

High-pass filter

To get high-pass biquadratic filter we have to adjust the elements as follows

$$Y_1 = sC_1, Y_2 = G_2, Y_3 = sC_3, Y_4 = sC_4, Y_5 = G_5,$$

The transfer function in such case is of the form

$$H_{HP}(s) = \frac{s \frac{C_1}{C_4}}{s^2 + \frac{C_1 + C_3 + C_4}{C_3 C_4} G_5 + \frac{G_2 G_5}{C_3 C_4}} \quad (2.50)$$

Band-pass filter

Band-pass filter can be realized in multiloop filter structure by assuming

$$Y_1 = G_1, Y_2 = G_2, Y_3 = sC_3, Y_4 = sC_4, Y_5 = G_5,$$

In such case the transfer function is given by

$$H_{BP}(s) = \frac{-s \frac{G_1}{C_4}}{s^2 + \frac{C_3+C_4}{C_3C_4} G_5 + \frac{G_5(G_1+G_2)}{C_3C_4}} \quad (2.51)$$

The design of particular biquadratic filter on the basis of the assumed specifications is transformed to solution of the set of equations comparing the needed form of denominator (eq. (2.44)-(2.46)) and the actual one (one of eq. (2.49)- (2.51)).

Example 2.7

Design biquadratic filter of the given below multiloop structure presented in Fig. 2.13 realizing BP characteristics of the required values of quality $Q=2$ and center frequency $f_0=1000\text{Hz}$. For the designed filter:

1.

Draw frequency (magnitude and phase) characteristics using Matlab.

2.

Estimate parameters (Q and f_0) on the basis of the magnitude and phase characteristics.

3.

Apply cascade connection of two and more identical biquadratic circuit structures and draw its frequency characteristics.

Solution

The band-pass filter should apply the following elements:

$$Y_1 = G_1, Y_2 = G_2, Y_3 = sC_3, Y_4 = sC_4, Y_5 = G_5,$$

The design starts from normalized filter of $\omega_0=1$. Transfer function is defined in the form given below, in which the gain k is of no importance in our consideration

$$H_{BP}(s) = \frac{ks}{s^2 + 0.5s + 1}$$

Comparing the coefficients of this denominator and its general form given by (2.51) leads to the system of 2 equations

$$G_5/C_4 + G_5/C_3 = 0.5$$

$$G_5(G_1+G_2)/(C_3C_4) = 1$$

They represent 2 equations in 5 variables, therefore, we may expect many different solutions. One of them, assumed by us is of the form

$$G_1=8S, G_2=8S, C_3=4F, C_4=4F, G_5=1S$$

They are in the normalized form. To find proper real values of elements we have assumed $k_r=10^4$ and $k_f=2\pi 10^3$. Applying it we get the real values of the elements applied in practical implementation of filter:

$$R_{1r} = \frac{1}{8} 10^4 = 1250\Omega$$

$$R_{2r} = R_{1r} = 1250\Omega$$

$$R_{5r} = 10^4 = 10000\Omega$$

$$C_{3r} = 4 \frac{1}{10^4 2\pi 10^3} = 63.662 nF$$

$$C_{4r} = C_{3r} = 63.662 nF$$

The program, shown below, applies these values of elements in simulation of the filter in Matlab.

```
% Program in Matlab for BP filter
% Real parameters of filter
kr=10000, kf=2*pi*1000;
g1=1/1250; g2=g1; g5=1/10000; c3=4/(kr*kf); c4=c3;
% Frequency characteristics
omega=0:5:20000;
lo=length(omega)
for k=1:lo
h(k)=g1/c4*j*omega(k)/(-omega(k)^2 +j*omega(k)*(g5/c4+g5/c3)+g5*(g1+g2)/c3/c4);
end
figure(1)
subplot(2,1,1), plot(omega, abs(h)/max(abs(h))), grid, ylabel('magnitude')
subplot(2,1,2), plot(omega, angle(h)), grid, ylabel('phase')
% Cascade connection of 2 identical sections
h2=h.^2;
figure(2)
subplot(2,1,1), plot(omega, abs(h2)/max(abs(h2))), grid, ylabel('magnitude')
subplot(2,1,2), plot(omega, angle(h2)), grid, ylabel('phase')
% Cascade connection of 4 identical sections
h4=h.^4;
figure(3)
subplot(2,1,1), plot(omega, abs(h4)/max(abs(h4))), grid, ylabel('magnitude')
subplot(2,1,2), plot(omega, angle(h4)), grid, ylabel('phase')
```

The graphical result presenting frequency characteristics of the designed biquadratic filter is presented in Fig. 2.15.

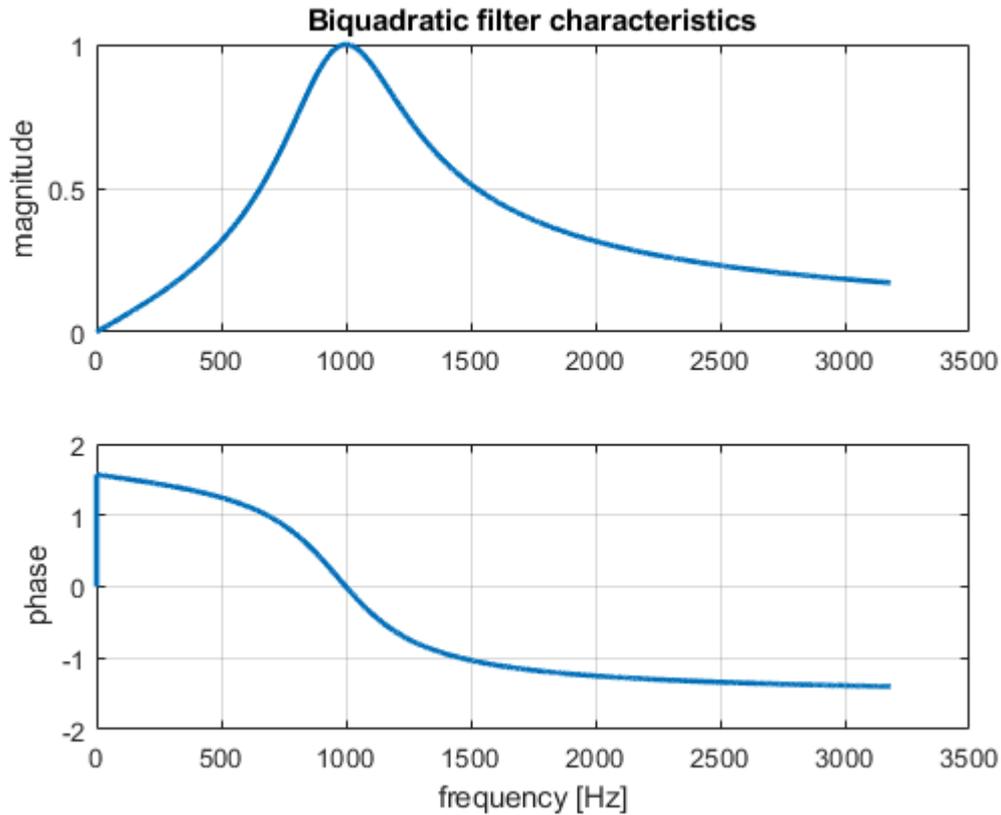


Fig. 2.15 Frequency characteristics of the designed band-pass biquadratic filter.

2.8 Exercises

1. Transform low-pass (LP) filter of $H(s)=1/(s^2+s+1)$ into

1. LP of $\omega_{0r} = 1000$
2. HP of $\omega_{0r} = 1000$
3. BP of $\omega_{1r} = 1000$ and $\omega_{2r} = 2000$
4. BS of $\omega_{1r} = 1000$ and $\omega_{2r} = 2000$

Solution:

1. LP \rightarrow LP

$$H_{LP}(s) = \frac{1}{(s/1000)^2 + (s/1000) + 1} = \frac{10^6}{s^2 + 1000s + 10^6}$$

1. LP \rightarrow HP

$$H_{HP}(s) = \frac{1}{(1000/s)^2 + (1000/s) + 1} = \frac{s^2}{s^2 + 1000s + 10^6}$$

1. LP \rightarrow BP

$$\omega_{0r} = \sqrt{2 \cdot 10^6} = \sqrt{2} \cdot 1000 \quad , B=1000, \text{to } \frac{\{s^2 + \omega_{0r}^2\}}{B \cdot s}$$

$$H_{BP}(s) = \frac{1}{\left(\frac{s^2+2 \cdot 10^6}{1000s}\right)^2 + \left(\frac{s^2+2 \cdot 10^6}{1000s}\right) + 1} = \frac{10^6 s^2}{s^4 + 1000s^3 + 5 \cdot 10^6 s^2 + 2 \cdot 10^9 s + 4 \cdot 10^{12}}$$

1. LP \rightarrow BS

$$\omega_{0r} = \sqrt{2 \cdot 10^6} = \sqrt{2} \cdot 1000 \quad , B=1000, \text{to } \frac{B \cdot s}{\{s^2 + \omega_{0r}^2\}}$$

$$H_{BS}(s) = \frac{1}{\left(\frac{1000s}{s^2+2 \cdot 10^6}\right)^2 + \left(\frac{1000s}{s^2+2 \cdot 10^6}\right) + 1} = \frac{s^4 + 4 \cdot 10^6 s^2 + 4 \cdot 10^{12}}{s^4 + 1000s^3 + 5 \cdot 10^6 s^2 + 2 \cdot 10^9 s + 4 \cdot 10^{12}}$$

1.

Design KHN band-pass filter of parameters:

$$Q=5, f_o=1000\text{Hz}, A_{BP}=1$$

Solution:

The KHN circuit is presented in Fig. 2.16

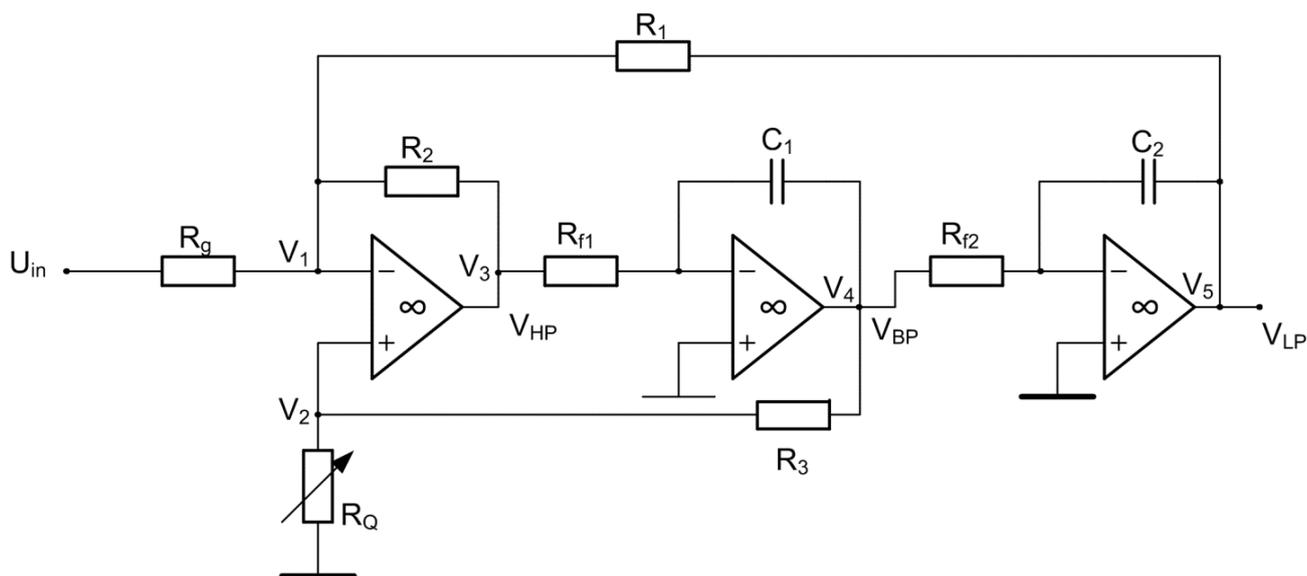


Fig. 2.16 KHN circuit structure

Assume $C_1=C_2=C=100\text{nF}$, $R_1=R_2=R_3=5\text{k}\Omega$ and $R_{f1}=R_{f2}=R_f$

Hence

$$R_f = \frac{1}{2\pi f_o C} = 1.59\text{k}\Omega$$

$$R_g = \frac{RQ}{A_{BP}} = 25\text{k}\Omega$$

$$R_Q = \frac{R}{2Q + \frac{QR}{R_g} - 1} = 0.5k\Omega$$

1. Generate transfer function of Butterworth, Chebyshev and elliptic filters of 4th order. Assume normalized frequency $\omega_0 = 1$, $R_p=3\text{dB}$ and $R_s=40\text{dB}$. Plot frequency characteristics of all filters in common subplots. Apply Matlab.

Solution:

Matlab program generating transfer functions:

```
[Lb,Mb]=butter(4,1,'s')
[Lc,Mc]=cheby1(4,3,1,'s')
[Le,Me]=ellip(4,3,40,1,'s')
```

The results of this program:

```
Lb = 0 0 0 0 1.0000
Mb = 1.0000 2.6131 3.4142 2.6131 1.0000

Lc = 0 0 0 0 0.1253
Mc = 1.0000 0.5816 1.1691 0.4048 0.1770

Le = 0.0100 -0.0000 0.1101 -0.0000 0.1815
Me = 1.0000 0.5736 1.2654 0.4589 0.2564
```

Hence the transfer functions of these filters are as follows

1. Butterworth filter:

$$H_B(s) = \frac{1}{s^4 + 2.613s^3 + 3.414s^2 + 2.613s + 1}$$

1. Chebyshev filter:

$$H_C(s) = \frac{1}{s^4 + 0.581s^3 + 1.169s^2 + 0.405s + 0.177}$$

1. Elliptic filter:

$$H_E(s) = \frac{0.01s^4 + 0.110s^2 + 0.182}{s^4 + 0.574s^3 + 1.265s^2 + 0.459s + 0.256}$$

Matlab program plotting frequency characteristics:

```
w=0:0.01:pi;
```

```

hb=freqs(Lb,Mb,w);
hc=freqs(Lc,Mc,w);
he=freqs(Le,Me,w);
subplot(2,1,1), plot(w, abs(hb),'r-', w, abs(hc),'g-', ...
w, abs(he),'b-')
grid, xlabel('Omega'), ylabel('Magnitude')
legend('Butterworth','Chebyshev','Elliptic')
subplot(2,1,2)
plot(w, unwrap(angle(hb)),'r-', w, unwrap(angle(hc)), ...
'g-',w, unwrap(angle(he)),'b-')
grid, xlabel('Omega'), ylabel('Phase')

```

The resulting plot is shown in figure below.

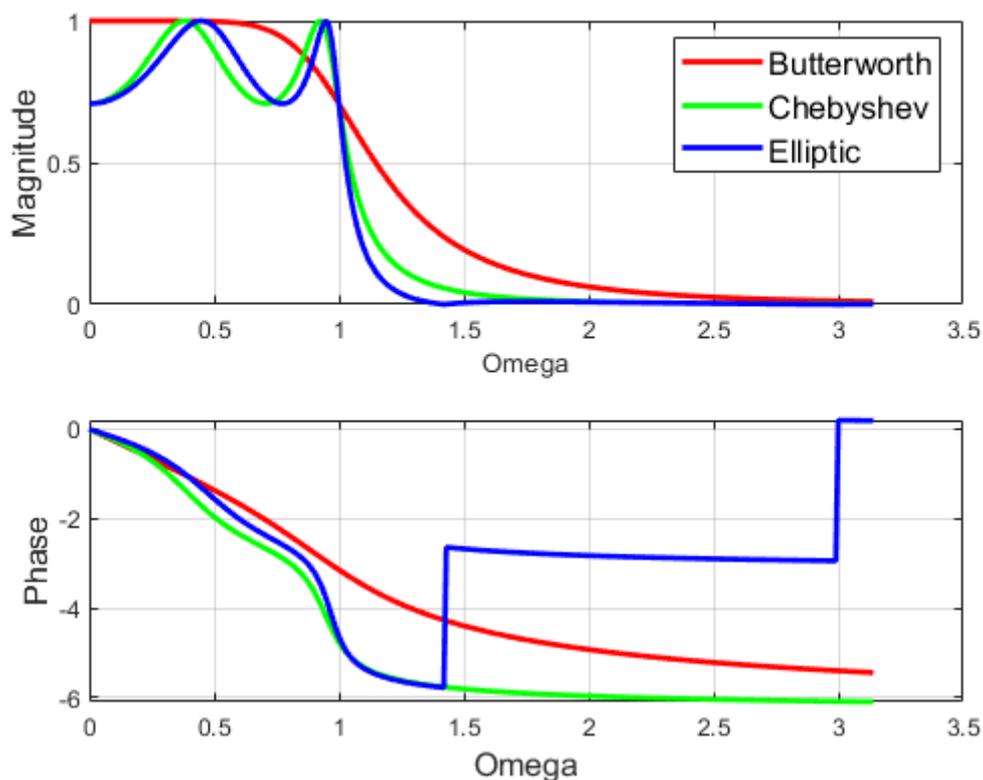


Fig. 2.17 The magnitude and phase characteristics of Butterworth, Chebyshev and elliptic filters of 4th order.

1.

Compare orders of Butterworth, Chebyshev and elliptic filters needed to design filters of the following specifications:

$\omega_p=0.3$, $\omega_s=0.35$, $R_p=3\text{dB}$ and $R_s=40\text{dB}$

Apply Matlab functions: *buttord*, *cheb1ord*, *ellipord*.

Solution:

Order of Butterworth filter: $N=25$

Order of Chebyshev filter: $N=9$

Order of elliptic filter: $N=5$

2.9 Basic definitions

Attenuation of filter – the inverse value of the gain of the magnitude characteristic. In logarithmic scale the attenuation is the negative values of the logarithmic gain.

Band-pass filter – filter that passes **frequencies** within a certain range and **attenuates** frequencies outside that range.

Band-stop filter – filter that attenuates **frequencies** within a certain range and **passes** frequencies outside that range.

Bandwidth – a measure of the width of a range of frequencies, measured in hertz.

Biquad – a second-order **linear filter**, containing two **poles** and in general two **zeros**, called also biquadratic filter.

Bode plot – the combination of the magnitude plot, expressing the magnitude (usually in **decibels**) of the frequency response, and a phase plot, expressing the **phase shift**.

Butterworth filter – type of **filter** designed to have a **frequency response** as flat as possible in the **passband** (usually referred to as a maximally flat magnitude filter).

Cascade of filters – the series of biquadratic filters connected in cascade fashion to reduce the sensitivity to parameter changes.

Chebyshev filter – filter having **passband** equiripple (type I) or **stopband** equiripple (type II) behavior. The magnitude of ripples is controlled by the ripple factor ϵ .

Chebyshev polynomial – a **sequence** of **orthogonal polynomials** which can be defined **recursively**. There are Chebyshev polynomials of degree n which represent the first kind denoted T_n and of the second kind which are denoted U_n .

Cutoff frequency – a boundary in a system's **frequency response** separating the pass band from stop band ranges of frequency.

Elliptic filter – filter with **equiripple** behavior in both the **passband** and the **stopband**. The magnitude of ripple is controlled by the ripple factor ϵ . It is also called Cauer filter.

Filter – device that removes some unwanted components or features from a **signal**. Here we understand it as the device passing or stopping signals of certain ranges of frequency.

Frequency characteristics – the change of transfer function $H(s=j\omega)$ with complex frequency $s=j\omega$. We recognize magnitude characteristics showing change of absolute values of characteristic with frequency and phase characteristic presenting change of angle with frequency.

Frequency scaling – the procedure of changing the cutoff frequency of filter from the normalized value (usually equal one) to the required, practical value.

High-pass filter – filter, that passes **signals** with a **frequency** higher than a certain **cutoff frequency** and **attenuates** signals with frequencies lower than the cutoff frequency.

KHN – biquadratic filter structure built on the basis of three operational amplifiers allowing independent fitting the quality and cutoff frequency of filter (named after their authors: Kervin-Huelsman-Newcomb).

Low-pass filter – **filter** that passes **signals** with a **frequency** lower than a selected **cutoff frequency** and **attenuates** signals with frequencies higher than the cutoff frequency.

Multiloop filter – the biquadratic filter built on the basis of one operational amplifier employing 2 feedback paths.

Normalized LP filter – the low-pass filter of the cutoff frequency equal one.

Passband – the range of frequencies in which the gain of the filter is higher than the assumed threshold (attenuation is lower than the assumed threshold).

Pole – the complex frequency s for which the transfer function $H(s)=N(s)/D(s)$ assumes infinite value. Usually they are the roots of $D(s)$.

Resistance scaling – the procedure of changing the resistance values of filter from the normalized value to the required, practical value (for example on the level of $k\Omega$).

Ripple factor – a **frequency domain** ripple that arises in **filter** magnitude characteristic. It means periodic variation in the **insertion loss** of the filter against increasing **frequency**. It is interpreted as maximum allowed attenuation ϵ in passband. For example $\epsilon=1$ corresponds to the attenuation equal 3dB for the angular frequency $\omega=\omega_0$).

Stopband – the range of frequencies in which the gain of the filter is lower than the assumed threshold (attenuation is higher than the assumed threshold).

Transit band – the range of frequencies, which belongs neither to passband nor to stopband.

Zero – complex frequency s for which the transfer function $H(s)=N(s)/D(s)$ assumes zero value. Usually zeros are the roots of $N(s)$.

4. Sensitivity of electrical circuits

Sensitivity analysis plays an important role in electrical engineering. It defines the measure how the response of the circuit is sensitive to the changes of values of elements. Observe, that all circuit components are manufactured around a target value of their main parameter (the rated value) with an allowable tolerance. Smaller tolerance requires higher technological accuracy, which involves the increasing of the component costs and vice versa. Therefore, the designer must decide which components are critical and what is the required value of the tolerance. Such a decision is possible only through a rigorous sensitivity analysis [4]. In this chapter we will study only small increments of the parameter values, hence, the considerations will be limited only to the first derivative of the output function.

4.1. Basic notions of sensitivity

Let us assume the variable x represent any parameter of the circuit, for example resistance, conductance, impedance, capacitance or inductance. Let $F(x)$ denotes the arbitrary response of the circuit, for example current or voltage. Different definitions of sensitivity might be applied [4,16].

Absolute (unnormalized) sensitivity

$$B_x^F = \frac{\partial F}{\partial x} \quad (4.1)$$

Relative sensitivity

$$S_x^F = \frac{\partial \ln(F)}{\partial \ln x} = \frac{\partial F}{\partial x} \frac{x}{F} \quad (4.2)$$

Semi-relative sensitivity

$$Q_x^F = \frac{\partial F}{\partial x} x \quad (4.3)$$

$$\text{\textbf{\textit{latex}}} P_{\{x\}^F} = \frac{\{\partial F\}}{\{\partial x\}} \frac{1}{F} \text{\textit{}} \quad (4.4)$$

The relative sensitivity represents the normalized value of the sensitivity. We may interpret it as the ration of the fractional change of the network function F to the fractional change in the parameter x , provided that all changes are sufficiently small (approaching zero theoretically). It is more convenient, because allows comparing the sensitivities with respect to many circuit parameters. If we know the sensitivity value at any operating point we are able to estimate the increment ΔF of the circuit as a result of small perturbation Δx of the parameter x . Especially important is the relative sensitivity. Using it we can calculate the expected change of the response F , namely

$$\frac{\Delta F}{F} = S_x^F \frac{\Delta x}{x} \quad (4.5)$$

The relative change of the response is proportional to the relative change of parameter x multiplied by the relative sensitivity value. The higher this sensitivity the larger change of the response.

For example $S_x^F = 1/2$ means that changing x by α results in the relative change of response equal $1/2\alpha$. Many different properties of sensitivity function can be observed. Some of them are presented in Table 4.1 [16].

Table 4.1. Basic properties of relative sensitivity.

Function $F(x)$	Relative sensitivity S_x^F
$F = F(\alpha(x))$	$S_x^F = S_\alpha^F S_x^\alpha$
$F = F(x)e^{j\phi(x)}$	$S_x^F = S_x^F + jQ_x^\phi$
$F = F_1(x)F_2(x)$	$S_x^F = S_x^{F_1} + S_x^{F_2}$
$F = F_1(x)/F_2(x)$	$S_x^F = S_x^{F_1} - S_x^{F_2}$
$F = (F_1(x))^a$	$S_x^F = aS_x^{F_1}$
$F = F(y), y = 1/x$	$S_x^F = -S_y^F$
$F = aF_1(x)$	$S_x^F = S_x^{F_1}$

These properties can simplify the sensitivity computation of the complex function $F(x)$. This will be illustrated in the following example.

Example 4.1

Let us consider function $F(x)$ given explicitly in the form

$$F(x) = \sqrt{\frac{x^2 - 3x + 5}{4x^3 - 6x}}$$

Introducing notation $F_1(x) = (x^2 - 3x + 5)^{0,5}$ and $F_2(x) = (4x^3 - 6x)^{0,5}$ we can present sensitivity S_x^F as following

$$S_x^F = S_x^{F_1} - S_x^{F_2} = 0,5 \left(S_x^{W_1(x)} + S_x^{W_1(x)} \right)$$

in which $W_1(x) = (x^2 - 3x + 5)$ and $W_2(x) = (4x^3 - 6x)$. After making proper calculations we get finally

$$S_x^F = 0,5 \left(\frac{2x - 3}{x^2 - 3x + 5} x + \frac{12x^2 - 6}{4x^3 - 6x} x \right)$$

4.2. Invariance relations

The sensitivity relations of the response to different parameters are dependent to each other and are in strict relations. For example the impedance and admittance functions are in inverse relation $Z=1/Y$, hence, we have

$$S_x^{Z(x)} = -S_x^{Y(x)} \quad (4.6)$$

Similarly, according to the properties of sensitivity expressed in Table 4.1 we observe

$$S_Z^F = -S_Y^F \quad (4.7)$$

Similarly, sensitivity of any function F with respect to the impedance of the inductor $Z_L=sL$ and to inductance L is the same. The same property is valid for the admittance $Y_C=sC$ and capacitance of the capacitor. Therefore, we have

$$S_{Z_L}^F = S_L^F \quad (4.8)$$

$$S_{Y_C}^F = S_C^F \quad (4.9)$$

Moreover, we can observe some interesting invariance relations of sensitivity to the admittances or impedances of the elements, irrespective of their type. Observe that the electrical elements may be described either by impedance or admittance relations. For the resistor we have $Z_R=R$ or $Y_R=1/R$, for inductor $Z_L=sL$ or $Y_L=1/sL$, for capacitor $Y_C=sC$ or $Z_C=1/sC$, for voltage controlled current source $I_s=gU_s$ (g – parameter of the admittance type), for current controlled voltage source $U_s=rI_s$ (r – parameter of the impedance type). For these type of elements the following invariance relations are valid

The sum of relative sensitivities of the voltage transfer function $T_v(s)$ with respect to all admittance or all impedance type parameters is zero. This can be written in the form

$$\sum_{Z_i} S_{Z_i}^{T_v(s)} = \sum_{Y_i} S_{Y_i}^{T_v(s)} = 0 \quad (4.10)$$

The sum of relative sensitivities of the arbitrary impedance function $Z(s)$ of the circuit (for example the input impedance) with respect to all impedance type parameters is equal 1 and with respect to all admittance type parameters equal -1. This can be written in the form

$$\begin{aligned} \sum_{Z_i} S_{Z_i}^{Z(s)} &= 1 \\ \sum_{Y_i} S_{Y_i}^{Z(s)} &= -1 \end{aligned} \quad (4.11)$$

In a similar way we can define the property regarding any admittance function. The sum of relative sensitivities of any admittance function $Y(s)$ of the circuit with respect to all impedance type parameters is equal -1 and with respect to all admittance type parameters equal 1.

$$\begin{aligned} \sum_{Z_i} S_{Z_i}^{Y(s)} &= -1 \\ \sum_{Y_i} S_{Y_i}^{Y(s)} &= 1 \end{aligned} \quad (4.12)$$

The sum of relative sensitivities of the arbitrary response function $F(s)$ of the circuit (for example the impedance $F(s)=Z(s)$, admittance $F(s)=Y(s)$ or any other transfer function $F(s)=T(s)$) with respect to all inductances and all capacitances is equal to the sensitivity to the complex frequency s , which can be written in the way

$$\sum_{L_i} S_{L_i}^{F(s)} + \sum_{C_i} S_{C_i}^{F(s)} = S_s^{F(s)} \quad (4.13)$$

Example 4.2

The invariance properties of the circuit will be illustrated now on the example of the circuit shown in Fig. 4.1

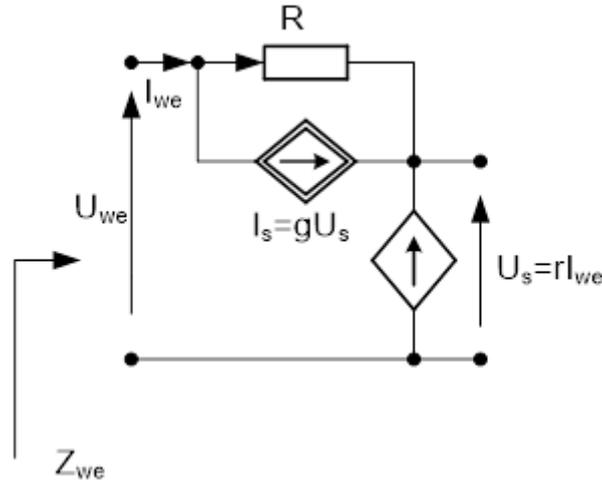


Fig. 4.1. The circuit subject to sensitivity analysis.

The circuit is described by the Kirchhoff voltage equation, from which we obtain the explicit form for the input impedance Z_{we}

$$U_{we} = R(I_{we} - grI_{we}) + rI_{we} \rightarrow Z_{we} = R + r - grR$$

Applying the differentiation formulas for this explicit expression of Z_{we} we can immediately write the sensitivity relations

$$\begin{aligned} S_R^{Z_{we}} &= (1 - gr) \frac{R}{Z_{we}} \\ S_r^{Z_{we}} &= (1 - gR) \frac{r}{Z_{we}} \\ S_g^{Z_{we}} &= -rR \frac{g}{Z_{we}} \end{aligned}$$

It is easy to find that the sensitivity invariance to all impedance type parameters is fulfilled, since

$$S_R^{Z_{we}} + S_r^{Z_{we}} - S_g^{Z_{we}} = \frac{R - Rgr + r - Rgr + Rgr}{Z_{we}} = 1$$

In a similar way we can check the sensitivity properties regarding voltage transfer function $T_v = U_s/U_{we}$. This voltage transfer function is equal

$$T_v = \frac{rI_{we}}{(r + R - rgR) I_{we}} = \frac{r}{r + R - rgR}$$

The sensitivity relations of this function with respect to parameters of the circuit are given as follows

$$\begin{aligned} S_r^{T_v} &= \frac{R}{r + R - rgR} \\ S_g^{T_v} &= \frac{rgR}{r + R - rgR} \\ S_R^{T_v} &= -\frac{R(1 - rg)}{r + R - rgR} \end{aligned}$$

It is easy to check, that $S_R^{T_v} + S_r^{T_v} - S_g^{T_v} = 0$.

The interesting sensitivity invariance properties can be defined also for RC circuits, containing no inductors.

The relative sensitivity of voltage transfer function $T_v(s)$ of the RC circuit with respect to all resistances and capacitances satisfies relations

$$\begin{aligned} \sum_{R_i} S_{R_i}^{T_v} &= \sum_{C_i} S_{C_i}^{T_v} = S_s^{T_v} \\ \sum_{R_i} S_{R_i}^{T_v} + \sum_{C_i} S_{C_i}^{T_v} &= 2S_s^{T_v} \end{aligned} \tag{4.14}$$

The sum of relative sensitivities of the impedance function $Z(s)$ or admittance function $Y(s)$ with respect to all capacitances is equal to the sensitivity to the complex frequency s

$$\begin{aligned} \sum_{C_i} S_{C_i}^{Z(s)} &= S_s^{Z(s)} \\ \sum_{C_i} S_{C_i}^{Y(s)} &= S_s^{Y(s)} \end{aligned} \tag{4.15}$$

Example 4.3

Let us consider the RC circuit presented in Fig. 4.2.

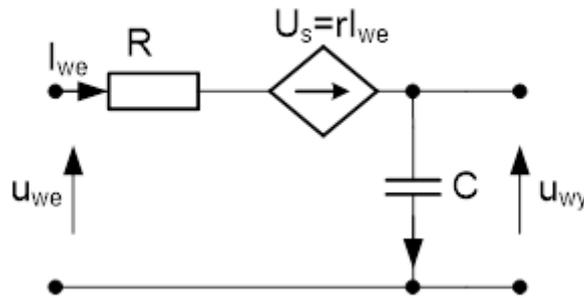


Fig. 4.2. The RC circuit subject to sensitivity analysis.

It is easy to show, that voltage transfer function $T_v(s)=U_{wy}/U_{we}$ can be described by the following relation

$$T_v(s) = \frac{1/sC}{R - r + 1/sC} = \frac{1}{(R - r)sC + 1}$$

After simple differentiation of this function we get the following relative sensitivity functions

$$\begin{aligned} S_R^{T_v(s)} &= \frac{-sCR}{(R-r)sC+1} \\ S_r^{T_v(s)} &= \frac{sCr}{(R-r)sC+1} \\ S_C^{T_v(s)} &= \frac{-(R-r)sC}{(R-r)sC+1} \end{aligned}$$

The invariance expressed by eq. (4.14) is given now in the form

$$\begin{aligned}\sum_{R_i} S_{R_i}^{T_v(s)} &= S_R^{T_v(s)} + S_r^{T_v(s)} = \frac{(r-R)sC}{(R-r)sC+1} \\ \sum_{C_i} S_{C_i}^{T_v(s)} &= S_C^{T_v(s)} = \frac{(r-R)sC}{(R-r)sC+1} \\ S_s^{T_v(s)} &= \frac{(r-R)sC}{(R-r)sC+1}\end{aligned}$$

In the same way we can define the sensitivity regarding input impedance $Z_{we}(s)=U_{we}/I_{we}$ of the circuit

$$Z_{we}(s) = R - r + 1/sC$$

After differentiation of this expression we get

$$\begin{aligned}S_C^{Z_{we}(s)} &= \frac{-1/sC}{R-r+1/sC} \\ S_s^{Z_{we}(s)} &= \frac{-1/sC}{R-r+1/sC}\end{aligned}$$

$$\sum_{C_i} S_{C_i}^{Z(s)} = S_s^{Z(s)}$$

It is easy to check the validity of the invariance relation (4.15) since

4.3. Adjoint network approach to sensitivity

Direct differentiation of the function is applicable only in the case when this function is given in an explicit form. However, in electrical circuits is a rare situation, since such relations are very difficult to obtain for large size networks. Instead different approaches, which do not require the explicit form of solution, have been developed. In this chapter we will present two such approaches. One is based on the idea of adjoint network and the second on incremental network. This section will be devoted to adjoint network method [4,7,16].

4.3.1. Tellegen theorem

Consider two circuits N and \bar{N} of identical topologies. Let us define vectors of corresponding branch currents in both circuits of the same directions. Apply the same for voltages. According to Tellegen theorem the following identities are valid [4,16]

$$\mathbf{U}^T \bar{\mathbf{I}} = \mathbf{I}^T \bar{\mathbf{U}} = \bar{\mathbf{U}}^T \mathbf{I} = \bar{\mathbf{I}}^T \mathbf{U} = 0 \quad (4.16)$$

The vectors \mathbf{U}, \mathbf{I} are applicable in circuit N , and $\bar{\mathbf{U}}, \bar{\mathbf{I}}$ in circuit \bar{N} . The superscript T means conjugate transposition of the complex vectors. Equation (4.16) does not have the physical meaning of “sum of branch powers”, since vectors \mathbf{V} and \mathbf{I} in general belong to two different networks.

Tellegen equation is valid for instantaneous values, transformed variables and complex representation of variables. The only significant condition required by the theorem is the identical topology of both circuit structures and not the type of elements (the corresponding branches in both circuits may be absolutely different, for example resistor in N and diode in \bar{N}). Observe that by adding some additional branches as short circuit or open circuit in the in adjoint network we can create the same topology, always satisfying the Tellegen condition.

4.3.2 Adjoint network

Two linear time-invariant circuits N and \hat{N} are said to be adjoint of each other if they satisfy the following conditions

Both circuits have the same topology, which means that their incidence matrices are the same

The hybrid matrices of both circuits satisfy the condition

$$\begin{bmatrix} \hat{\mathbf{H}}_{11} & \hat{\mathbf{H}}_{12} \\ \hat{\mathbf{H}}_{21} & \hat{\mathbf{H}}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{11}^T & -\mathbf{H}_{21}^T \\ -\mathbf{H}_{12}^T & \mathbf{H}_{22}^T \end{bmatrix} \quad (4.17)$$

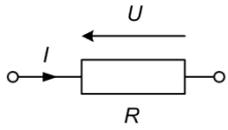
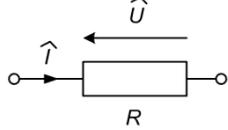
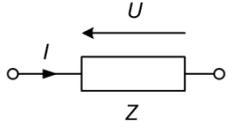
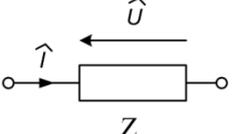
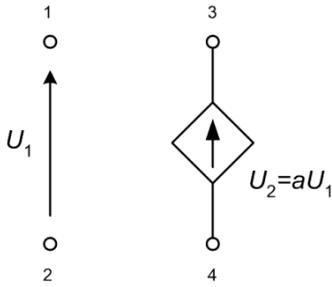
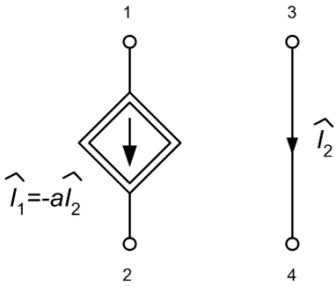
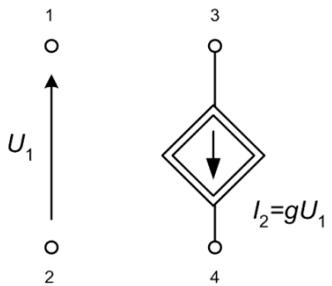
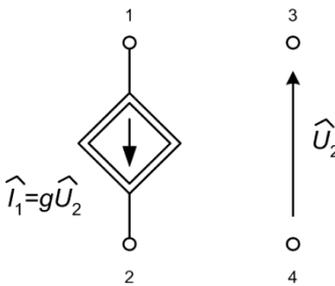
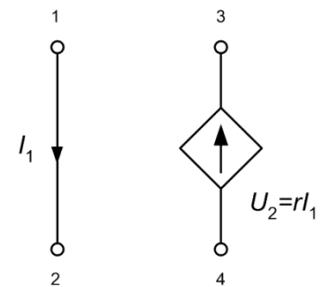
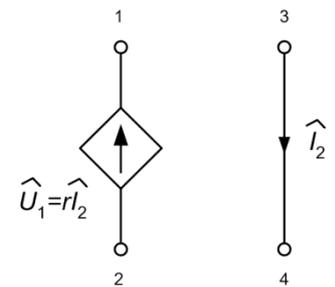
where the hybrid description is defined as follows

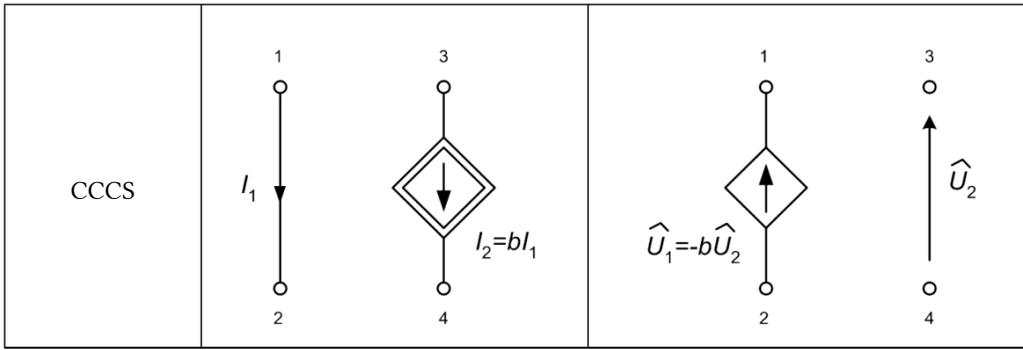
$$\begin{bmatrix} \mathbf{U}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{U}_2 \end{bmatrix} \quad (4.18)$$

Applying these relations we can define the adjoint version of any circuit elements. In this way the whole network, adjoint to the original one, can be built.

For example the 2-terminal element described by the impedance Z of the description $U=ZI$ after applying eq. (4.17) is transformed to its adjoint version, which is described by the same type of equation $\hat{U} = Z\hat{I}$. It mean that any passive 2-terminal element (resistor $Z=R$, inductor $Z=sL$, capacitor $Z=1/sC$) has the identical form as the original one. The first two rows of table 4.2 depict this situation.

Table 4.2 The set of original adjoint elements

Element description	Original element	Adjoint element
Resistor		
Impedance		
VCVS		
VCCS		
CCVS		



In the case of controlled sources the description is in the form of 2-dimensional hybrid matrix. As an example let us consider voltage controlled voltage source (VCVS). Its nominal description is as follows

$$\begin{bmatrix} I_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ I_2 \end{bmatrix}$$

Taking into account the condition (4.17) we get the following description of the adjoint element

$$\begin{bmatrix} \hat{I}_1 \\ \hat{U}_2 \end{bmatrix} = \begin{bmatrix} 0 & -a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{U}_1 \\ \hat{I}_2 \end{bmatrix}$$

It is evident that adjoint element of VCVS is the current source \hat{I}_1 controlled by current \hat{I}_2 , of the controlling coefficient equal $-a$, that is $\hat{I}_1 = -a\hat{I}_2$ with the proper exchange of the places of controlled and controlling currents. Table 4.2 depicts all types of controlled sources and their corresponding adjoint elements.

Observe, that passive elements R, L, C have adjoint counterparts of the same form as their originals. Therefore, the adjoint circuit to the passive RLC original one is exactly the same, with the only changes of notations of currents and voltages. It is in contrast to the circuits with controlled sources. Creation of adjoint circuit for controlled sources needs building the individual representation of each controlled source. The important point, that should be noted, is the directions of currents in both circuits should be opposite to the direction of voltages.

The rules of creating the adjoint networks are not applicable to the independent sources appearing in original circuit. They should be simply eliminated in the adjoint circuit (original voltage source terminal short circuit and current source terminal open).

Example 4.4

Let us consider the original circuit with controlled sources presented in Fig. 4.3a.

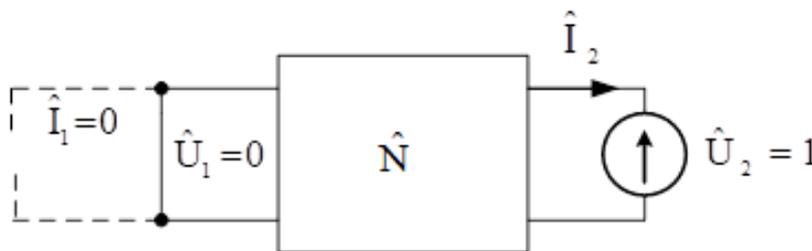


Fig. 4.5. Illustration of the terminals of circuits in computation of sensitivity of current I_2 : a) original circuit supplied from voltage (continuous) or current source (dashed line), b) the corresponding adjoint circuit supplied from the voltage source of unity value.

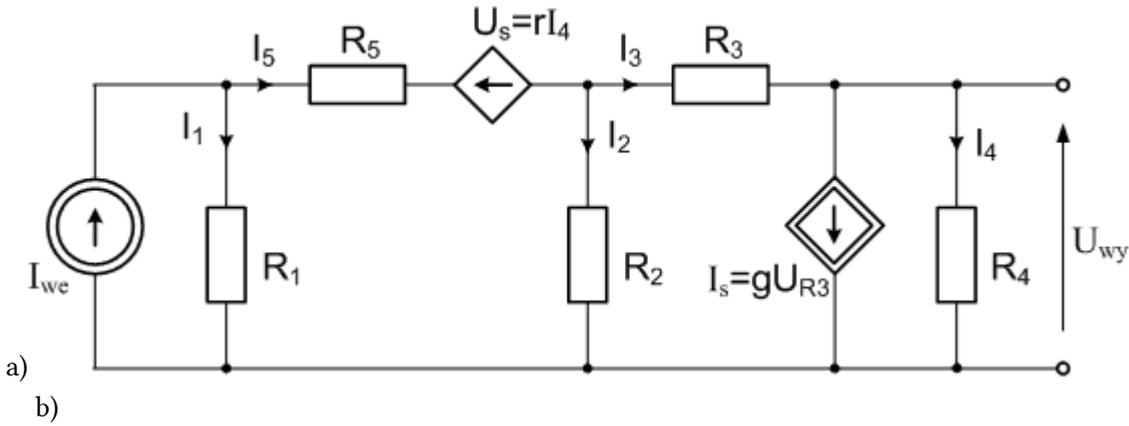


Fig. 4.3 An example of circuit with controlled sources: a) original circuit, b) adjoint circuit.

Following the rules of substituting the original elements by their adjoint counterparts allows building the structure of the adjoint circuit shown in Fig. 4.3b. The independent current source has been eliminated in adjoint circuit.

4.3.3. Tellegen theorem for incremental circuit

Let us consider multiport (original and adjoint) containing p external ports (the terminals representing either inputs or outputs) and internal elements (either passive RLC elements or controlled sources). In both parts of circuit the direction of current is opposite to direction of voltage (for passive and active elements in the same way). The vectors of voltages and currents will be distinguished by the indices: p for ports and b for other internal elements.

In such case the Tellegen theorem $\bar{\mathbf{U}}^T \mathbf{I} = \bar{\mathbf{I}}^T \mathbf{U} = 0$ may be written in the extended form, where the sign $\hat{}$ represents the adjoint network

$$\hat{\mathbf{I}}_p^T \mathbf{U}_p + \hat{\mathbf{I}}_b^T \mathbf{U}_b = \hat{\mathbf{U}}_p^T \mathbf{I}_p + \hat{\mathbf{U}}_b^T \mathbf{I}_b \quad (4.19)$$

Let us group the variables representing ports (index p) and internal branches (index b) on two sides of the equation. In such case we get

$$\hat{\mathbf{U}}_p^T \mathbf{I}_p - \hat{\mathbf{I}}_p^T \mathbf{U}_p = \hat{\mathbf{I}}_b^T \mathbf{U}_b - \hat{\mathbf{U}}_b^T \mathbf{I}_b \quad (4.20)$$

Substituting the original variables $\mathbf{U}_p, \mathbf{I}_p, \mathbf{U}_b, \mathbf{I}_b$ by their appropriate increments does not change anything in Tellegen description. Therefore we get the following form of Tellegen theorem for incremental structure [4,16]

$$\hat{\mathbf{U}}_p^T \Delta \mathbf{I}_p - \hat{\mathbf{I}}_p^T \Delta \mathbf{U}_p = \hat{\mathbf{I}}_b^T \Delta \mathbf{U}_b - \hat{\mathbf{U}}_b^T \Delta \mathbf{I}_b \quad (4.21)$$

This equation will be the basis for getting the sensitivity relations of the port variables denoted by index p (either voltages or currents) with respect to the parameters of the internal elements denoted by index b (the passive RLC elements and coefficients of the controlled sources).

4.3.4. Sensitivity relations

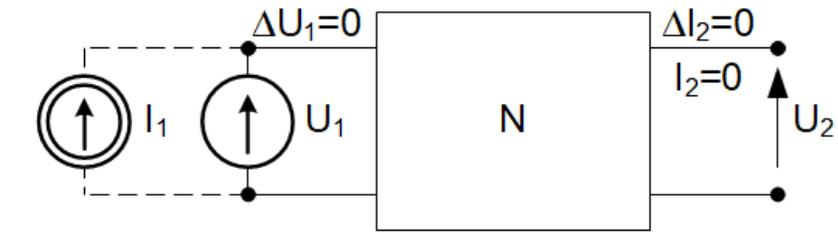
The equation (4.21) defines changes of the output variable, either currents $\Delta \mathbf{I}_p$ or voltages $\Delta \mathbf{U}_p$ on the output ports to the changes of parameters of the internal elements represented by $\Delta \mathbf{U}_b$ for voltages or $\Delta \mathbf{I}_b$ of currents of elements. Usually, we need to calculate sensitivity of one response (one element of vector \mathbf{U}_p or \mathbf{I}_p) to the changes of one or more elements of circuits. These elements are strictly associated with the vectors \mathbf{U}_b or \mathbf{I}_b .

The procedure should be organized in such a way that the left side of equations (4.21) should contain only this variable which is of interest in sensitivity analysis (either particular port voltage U_2 or port current I_2). Let us assume that the circuit is supplied from only one source (voltage U_1 or current I_1). In this way the circuit may be presented as the 2-port of the terminal variables U_1, I_1 on the input side and U_2, I_2 on the output side. Therefore, the left side of eq. (4.21) may be represented as follows

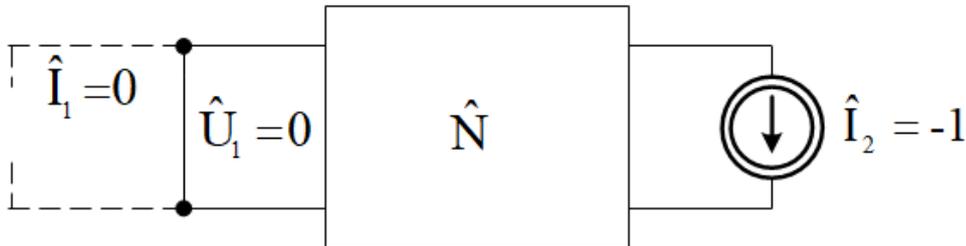
$$\hat{\mathbf{U}}_p^T \Delta \mathbf{I}_p - \hat{\mathbf{i}}_p^T \Delta \mathbf{U}_p = [\hat{U}_1 \quad \hat{U}_2] \begin{bmatrix} \Delta I_1 \\ \Delta I_2 \end{bmatrix} - [\hat{I}_1 \quad \hat{I}_2] \begin{bmatrix} \Delta U_1 \\ \Delta U_2 \end{bmatrix} \quad (4.22)$$

By proper choice of terminal conditions we should reduce the left side to ΔU_2 (when the sensitivity of voltage is required) or ΔI_2 (for sensitivity of current).

Let us first consider sensitivity of the output voltage U_2 . Additionally assume the supply in the form of either voltage source U_1 or current source I_1 , as is shown in symbolic way in Fig. 4.4a.



a)



b)

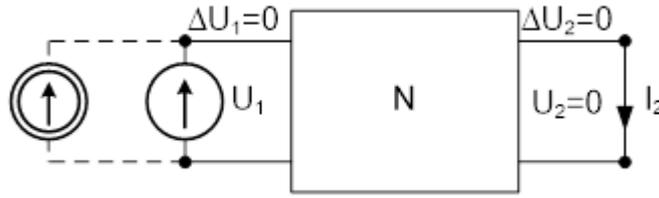
Fig. 4.4. Illustration of the terminals of circuits in computation of sensitivity of voltage U_2 : a) original circuit supplied from voltage (continuous) or current source (dashed line), b) the corresponding adjoint circuit supplied from the current source of unity value.

In the case of voltage source (U_1 of constant value) and lack of loading we have $\Delta U_1=0$, $I_2=0$ and $\Delta I_2=0$. To get the increment of only ΔU_2 in eq. (4.22) we should short circuit input terminals of adjoint circuit ($\hat{U}_1 = 0$) and assume current source as its source $\hat{I}_2 = -1$ of the direction opposite to the original voltage U_2 , as is shown in Fig. 4.4b.

In the case of current source (I_1 constant) supplying the original circuit of Fig. 4.4a we have $\Delta I_1=0$, $I_2=0$ and $\Delta I_2=0$. To get only increment ΔU_2 in eq. (4.22) we should open the terminals of former current source I_1 assuming $\hat{I}_1 = 0$ and assume the current excitation $\hat{I}_2 = -1$ (identically as in the previous case). Such situation is illustrated in Fig. 4.4b.

In a similar way we can propose the proper port terminations for adjoint circuit in the case of sensitivity of the output current I_2 . The original circuit in such case is presented in Fig. 4.5a.

a)



b)

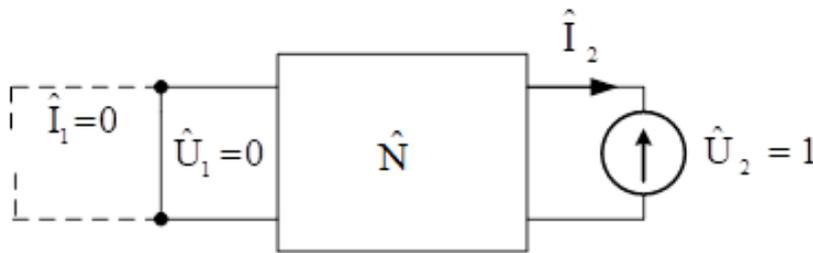


Fig. 4.5. Illustration of the terminals of circuits in computation of sensitivity of current I_2 : a) original circuit supplied from voltage (continuous) or current source (dashed line), b) the corresponding adjoint circuit supplied from the voltage source of unity value.

In the case of voltage supply in original circuit ($U_1 = \text{const}$) in Fig. 4.5a we have $\Delta U_1 = 0$, and $\Delta U_2 = 0$ (the output port short circuit). To obtain the increment ΔI_2 in eq. (4.22) we have to short circuit former input port ($\hat{U}_1 = 0$) and assume voltage excitation $\hat{U}_2 = 1$ of the direction opposite to the current I_2 in original circuit, as shown in Fig. 4.5b.

In the case of current excitation ($I_1 = \text{const}$) of Fig. 4.5a we have $\Delta I_1 = 0$ and also $\Delta U_2 = 0$. To obtain ΔI_2 in eq. (4.22) we should open input terminals of adjoint circuit ($\hat{I}_1 = 0$) and assume unit voltage excitation $\hat{U}_2 = 1$ (the same in the previous case). The final form of adjoint circuit is illustrated in Fig. 4.5b.

Observe that in any case the original sources disappear in its adjoint counterpart (current sources open and voltage sources short circuit). Instead, in adjoint circuit we apply only the unity sources (voltage source in calculation of sensitivity of output current and current source for sensitivity of output voltage). Finally we get the required increase of response, either ΔU_2 or ΔI_2 .

Let us consider now the right side of incremental Tellegen equation (4.21). It should be expressed as a function of the increments of parameters of the elements, with respect to which the sensitivity functions are calculated. In such case the increment of response (ΔF) will be expressed as a linear function of the increment

$$B_x^F = \frac{\Delta F}{\Delta x}$$

of the parameter (Δx). As a result we get the absolute sensitivity formula

Each element of the circuit needs special treatment. Let us start from the impedance Z , for which $U = ZI$ in original circuit and also $\hat{U} = Z\hat{I}$ in adjoint one. Therefore, the right side of eq. (4.21) takes the form

$$\hat{\mathbf{I}}_b^T \Delta \mathbf{U}_b - \hat{\mathbf{U}}_b^T \Delta \mathbf{I}_b = \hat{I} \left(\frac{\partial U}{\partial I} \Delta I + \frac{\partial U}{\partial Z} \Delta Z \right) - Z \hat{I} \Delta I \quad (4.23)$$

If we take into account that $\frac{\partial U}{\partial I} = Z$ and $\frac{\partial U}{\partial Z} = I$ the right side of eq. (4.23) is simplified to

$$\hat{\mathbf{I}}_b^T \Delta \mathbf{U}_b - \hat{\mathbf{U}}_b^T \Delta \mathbf{I}_b = I \hat{I} \Delta Z \quad (4.24)$$

From general equations (4.21) we get the sensitivity formula for impedance Z

$$B_Z^F = \frac{\Delta F}{\Delta Z} = I \hat{I} \quad (4.25)$$

This equation shows, that the derivative of the response F does not need to know the explicit form of it. It is enough to calculate the current of the impedance Z in the original (I) and adjoint circuit (\hat{I}) and multiply them.

In a similar way we can obtain the sensitivity formulas for any circuit element, including controlled sources. As an example let us take the voltage controlled voltage source (VCVS), for which we have $U_2 = aU_1$. Note, that for this type of source we have also $I_1 = 0$, $\Delta U_1 = 0$ (the controlling variable is assumed constant). From the

$$\Delta U_2 = \frac{\partial U_2}{\partial U_1} \Delta U_1 + \frac{\partial U_2}{\partial a} \Delta a$$

mathematic rules we have

. Additionally for the adjoint element we have

$\hat{U}_2 = 0$. The right side of eq. (4.21) takes now the form

$$\hat{\mathbf{I}}_b^T \Delta \mathbf{U}_b - \hat{\mathbf{U}}_b^T \Delta \mathbf{I}_b = \begin{bmatrix} \hat{I}_1 & \hat{I}_2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{\partial U_2}{\partial U_1} \Delta U_1 + \frac{\partial U_2}{\partial a} \Delta a \end{bmatrix} - \begin{bmatrix} \hat{U}_1 & \hat{U}_2 \end{bmatrix} \begin{bmatrix} 0 \\ \Delta I_2 \end{bmatrix} = \hat{I}_2 \frac{\partial U_2}{\partial a} \Delta a = U_1 \hat{I}_2 \Delta a$$

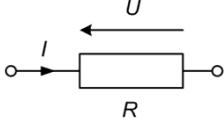
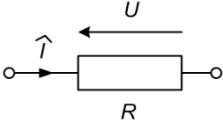
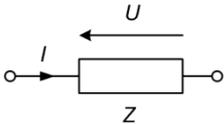
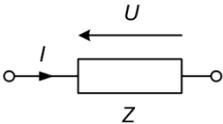
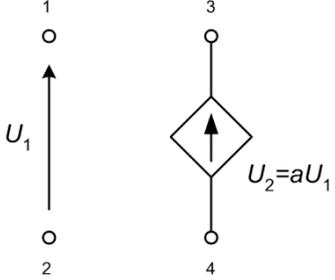
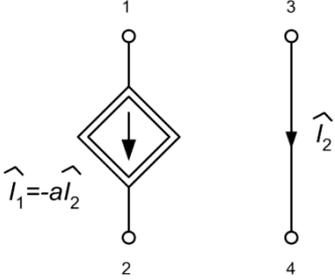
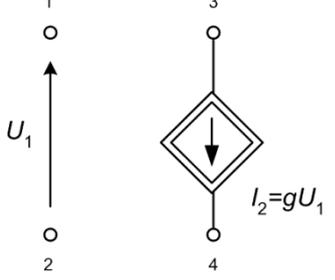
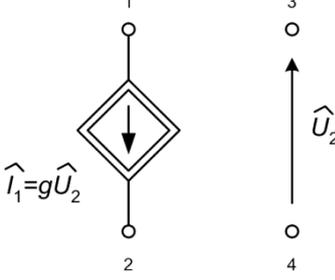
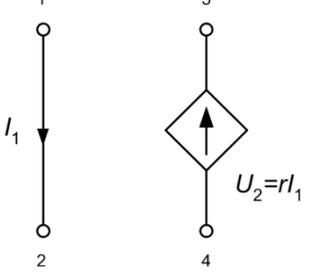
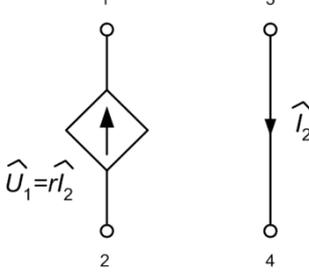
(4.26)

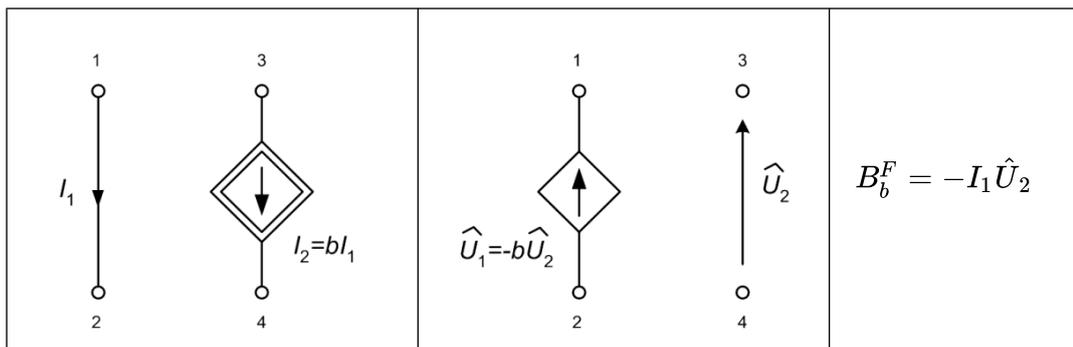
Hence the sensitivity formula for VCVS can be expressed now as follows

$$B_a^F = U_1 \hat{I}_2 \quad (4.27)$$

It is evident that sensitivity function is the product of two controlling variables: U_1 in original circuit and \hat{I}_2 in adjoint one. Table 4.2 depicts the sensitivity formulas (the third column) for the passive elements (the values R, G, Z, Y) and four types of controlled sources (the controlling coefficients).

Table 4.3 Sensitivity relations in adjoint network approach

Original element	Adjoint element	Sensitivity relation
		$B_R^F = I \cdot \hat{I}$ $B_G^F = -U \cdot \hat{U}$
		$B_Z^F = I \cdot \hat{I}$ $B_Y^F = -U \cdot \hat{U}$
		$B_a^F = U_1 \hat{I}_2$
		$B_g^F = -U_1 \hat{U}_2$
		$B_r^F = I_1 \hat{I}_2$



Observe, that obtaining the sensitivity values to many circuit elements needs to perform only two analyses: one of original circuit and the second of the adjoint circuit. On the basis of the results of such analyses the sensitivity values of the considered response with respect to all elements are determined.

Example 4.5

As an example let us determine the sensitivity of the input impedance Z_{we} of the circuit of Fig. 4.6a with respect to all resistances. Assume the values of resistances as follows: $R_1=20\Omega$, $R_2=10\Omega$, $R_3=5\Omega$, $R_4=5\Omega$.

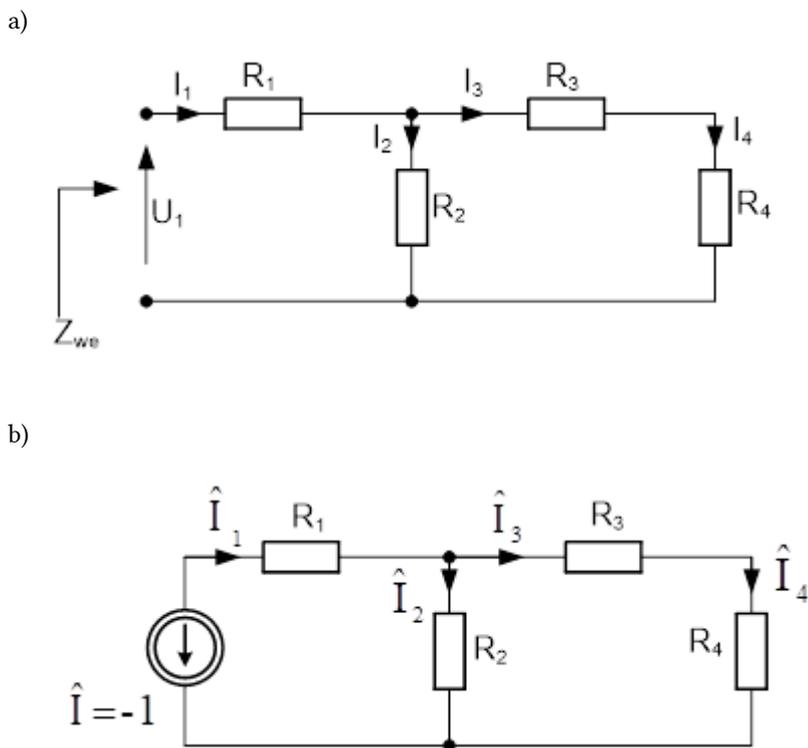


Fig. 4.6 The exemplary circuit in sensitivity analysis: (a) original, (b) adjoint

The input impedance of the circuit is defined as $Z_{we}=U_1/I_1$. To apply the adjoint approach to sensitivity computation we assume current excitation of unity value $I_1=1$. Therefore, $Z_{we}=U_1$ and we should apply procedure for sensitivity of voltage U_1 . The analysis of only two circuits is needed. For the original circuit we have the following Kirchhoff's equations and their solutions

$$\begin{aligned}
Z_{we} &= 20 + 5 = 25 \\
U_1 &= 25I_1 = 25 \rightarrow I_1 = 1 \\
U_2 &= 5I_1 \\
I_2 &= 5/10 = 0.5 \\
I_3 &= I_4 = 5/10 = 0.5
\end{aligned}$$

The adjoint circuit should be supplied by the current source $\hat{I}_1 = -1$. The structure of adjoint circuit is identical to the original circuit, since there are no controlled sources (Fig. 4.6b). Applying Kirchhoff's equations we get the solution as follows

$$\begin{aligned}
\hat{I}_1 &= I_1 = 1 \\
\hat{I}_2 &= 0.5 \\
\hat{I}_3 &= \hat{I}_4 = 0.5
\end{aligned}$$

On the basis of both solutions we calculate the sensitivity of input impedance as follows

$$\begin{aligned}
\frac{\partial Z_{we}}{\partial R_1} &= I_1 \hat{I}_1 = 1 \rightarrow S_{R_1}^{Z_{we}} = 1 \frac{20}{25} = 0.8 \\
\frac{\partial Z_{we}}{\partial R_2} &= I_2 \hat{I}_2 = 0.25 \rightarrow S_{R_2}^{Z_{we}} = 0,25 \frac{10}{25} = 0.1 \\
\frac{\partial Z_{we}}{\partial R_3} &= I_3 \hat{I}_3 = 0.25 \rightarrow S_{R_3}^{Z_{we}} = 0,25 \frac{5}{25} = 0.05 \\
\frac{\partial Z_{we}}{\partial R_4} &= I_4 \hat{I}_4 = 0.25 \rightarrow S_{R_4}^{Z_{we}} = 0,25 \frac{5}{25} = 0.05
\end{aligned}$$

$$\sum_{i=1}^4 S_{R_i}^{Z_{we}} = 1$$

It is easy to show that the basic invariance property holds, . This can be confirmed also by direct differentiation of the expression for input impedance Z_{we} , which is defined as follows

$$Z_{we} = R_1 + \frac{R_2(R_3 + R_4)}{R_2 + R_3 + R_4}$$

After proper differentiation we get

$$\begin{aligned}
\frac{\partial Z_{we}}{\partial R_1} &= 1 \\
\frac{\partial Z_{we}}{\partial R_2} &= \frac{(R_3 + R_4)(R_2 + R_3 + R_4) - R_2(R_3 + R_4)}{(R_2 + R_3 + R_4)^2} = 0.25 \\
\frac{\partial Z_{we}}{\partial R_3} &= \frac{R_2(R_2 + R_3 + R_4) - R_2(R_3 + R_4)}{(R_2 + R_3 + R_4)^2} = 0.25 \\
\frac{\partial Z_{we}}{\partial R_4} &= \frac{R_2(R_2 + R_3 + R_4) - R_2(R_3 + R_4)}{(R_2 + R_3 + R_4)^2} = 0.25
\end{aligned}$$

These results are the same as those obtained by adjoint network approach. It is worth to observe that exact result of differentiation of the network function was possible without knowing the explicit form of the function under differentiation. Moreover, all operations are done in numerical form, therefore, the method can be easily implemented in computer program.

Example 4.6

The next example will present the adjoint method for the circuit containing controlled sources (Fig. 4.7). The sensitivity of output voltage U_{wy} and input impedance with respect to all conductances G_i ($i = 1, 2, \dots$,

7) and controlling coefficients g_i ($i = 1, 2, 3$) of controlled sources will be calculated by using adjoint method implemented in the form of complete Matlab program.

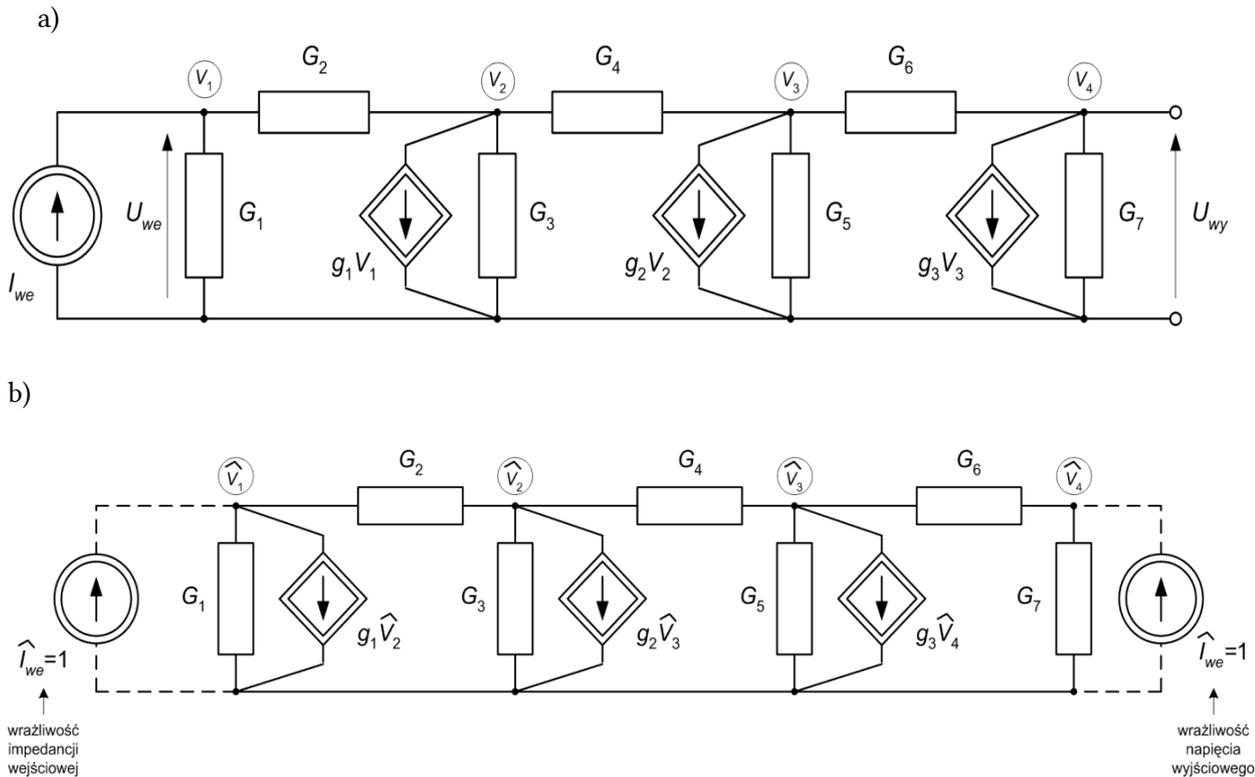


Fig. 4.7. Structures of the a) original and b) adjoint networks in sensitivity analysis.

The original circuit is excited by the unity current source. It means that input impedance is equal to the input voltage and output voltage equivalent to transfer function voltage-to-current.

Fig. 4.7b represents the adjoint circuit. The dashed lines indicate the possible excitation of this circuit. In calculation of sensitivity of output voltage we apply only the unity current source $\hat{I}_{we} = 1$ connected parallel to the conductance G_7 .

In the case of sensitivity of input impedance (input voltage) the excitation is applied at the input of circuit in the form of unity current source (identically as in original circuit). The Matlab file implementing all sensitivity computations is given below.

```
% Sensitivity of ladder network
% Elements
format compact
G1=1; G2=2; G3=1; G4=2; G5=1; G6=2; G7=1;
g1=1; g2=3; g3=1;Iwe=1;
% Analysis of original circuit – nodal method
Y=[G1+G2 -G2 0 0
  -G2+g1 G2+G3+G4 -G4 0
  0 -G4+g2 G4+G5+G6 -G6
  0 0 -G6+g3 G6+G7];
Iz=[Iwe; 0; 0; 0];
```

```

V=Y\Iz;
Uwy=V(4);
UG1=V(1); UG2=V(1)-V(2); UG3=V(2); UG4=V(2)-V(3); UG5=V(3); UG6=V(3)-V(4); UG7=V(4);
% Analysis of adjoint circuit
Yd=Y';
Izd=[0; 0; 0; 1];
Vd=Yd\Izd;
UGd1=Vd(1); UGd2=Vd(1)-Vd(2); UGd3=Vd(2); UGd4=Vd(2)-Vd(3);
UGd5=Vd(3); UGd6=Vd(3)-Vd(4); UGd7=Vd(4);
% Absolute sensitivity of the output voltage
BG1=-UG1*UGd1; BG2=-UG2*UGd2; BG3=-UG3*UGd3; BG4=-UG4*UGd4; BG5=-UG5*UGd5;
BG6=-UG6*UGd6; BG7=-UG7*UGd7; Bg1=-V(1)*Vd(2); Bg2=-V(2)*Vd(3); Bg3=-V(3)*Vd(4);
% Relative sensitivity
SUG1=BG1*G1/Uwy; SUG2=BG2*G2/Uwy; SUG3=BG3*G3/Uwy; SUG4=BG4*G4/Uwy; SUG5=BG5*G5/Uwy;
SUG6=BG6*G6/Uwy; SUG7=BG7*G7/Uwy; SUG1=Bg1*g1/Uwy; SUG2=Bg2*g2/Uwy; SUG3=Bg3*g3/Uwy;

% Analysis of adjoint circuit for input impedance
Izd=[1; 0; 0; 0];
Vd=Yd\Izd;
UGd1=Vd(1); UGd2=Vd(1)-Vd(2); UGd3=Vd(2); UGd4=Vd(2)-Vd(3);
UGd5=Vd(3); UGd6=Vd(3)-Vd(4); UGd7=Vd(4);
Zwe=V(1)/Iwe;
% Absolute sensitivity
BG1=-UG1*UGd1; BG2=-UG2*UGd2; BG3=-UG3*UGd3; BG4=-UG4*UGd4; BG5=-UG5*UGd5;
BG6=-UG6*UGd6; BG7=-UG7*UGd7; Bg1=-V(1)*Vd(2); Bg2=-V(2)*Vd(3); Bg3=-V(3)*Vd(4);
% Relative sensitivity
SZG1=BG1*G1/Zwe; SZG2=BG2*G2/Zwe; SZG3=BG3*G3/Zwe; SZG4=BG4*G4/Zwe; SZG5=BG5*G5/Zwe;
SZG6=BG6*G6/Zwe; SZG7=BG7*G7/Zwe; SZg1=Bg1*g1/Zwe; SZg2=Bg2*g2/Zwe; SZg3=Bg3*g3/Zwe

```

As a result of these computations of sensitivity of output voltage with respect to all parameters arranged as follows: $G_1, G_2, G_3, G_4, G_5, G_6, G_7, g_1, g_2, g_3$ we got

$$S_x^{Uwy} = [-0.3797 \ 1.2406 \ -0.2086 \ -2.7380 \ -0.2086 \ 1.2406 \ -0.3797 \ -1.1390 \ 2.7112 \ -1.1390]$$

It is easy to check that the invariance property of voltage-to-current transfer function (impedance type function) is fulfilled, since

$$SUG1+SUG2+SUG3+SUG4+SUG5+SUG6+SUG7+SUG1+SUG2+SUG3=-1$$

In the case of input impedance the following sensitivity values have been obtained)

$$S_x^{Zwe} = [-0.3797 \ -0.3932 \ -0.0255 \ -0.0337 \ 0.0027 \ 0.0012 \ 0.0006 \ -0.1390 \ -0.0352 \ 0.0018]$$

Also here the invariance property of the sensitivity is fulfilled, since

$$SZG1+SZG2+SZG3+SZG4+SZG5+SZG6+SZG7+SZg1+SZg2+SZg3=-1$$

4.3.5. Interpretation of adjoint method by SFG

It is well known, that both circuits: the original and the adjoint one can be represented by signal flow graphs.

Let us assume original circuit represented by graph G and its adjoint counterpart by \hat{G} . The graph \hat{G} is created as the transposition of G , that is by inverting the directions of the branches of original SFG, while keeping the same values of the branch gains. Observe, that in such inversion the source node of G is converted in \hat{G} to the sink node, and output node X_{wy} of G to the source node in \hat{G} . The source signal in \hat{G} should take the unity value. At such assumptions the sensitivity relations of the signal X_{wy} with respect to the gains w_{ij} of the branches of the graph take the form

$$\frac{dX_{wy}}{dw_{ij}} = \hat{X}_i X_j \quad (4.28)$$

All signals in the original graph are denoted by (X_j) and in adjoint graph by \hat{X}_i

. The parameter w_{ij} represents the gain of the branch starting from j th node and ending in i th node of original graph G (and in opposite directions in \hat{G}).

Let us sum up the procedure of application of SFG in sensitivity analysis.

For the given graph G of the source node signal X_{we} and output node signal X_{wy} create the adjoint graph \hat{G} in the following way.

All branches of \hat{G} exist also in G and are of the same gains but opposite directions.

The source node in \hat{G} is now the former output node. The excitation signal applied in this node has the unity value.

Analyze both graphs G and \hat{G} and calculate all their signals.

Calculate sensitivity of the output signal X_{wy} with respect to all gains w_{ij} as the product of signals from which the gain w_{ij} starts in original graph (X_j) and in adjoint graph (\hat{X}_i) applying the eq. (4.21).

Example 4.7

The procedure of application of SFG in sensitivity calculations will be illustrated on the example of graph presented in Fig. 4.7a. The gains of the branches are denoted as : w_1, w_2, w_3, w_4, w_5 .

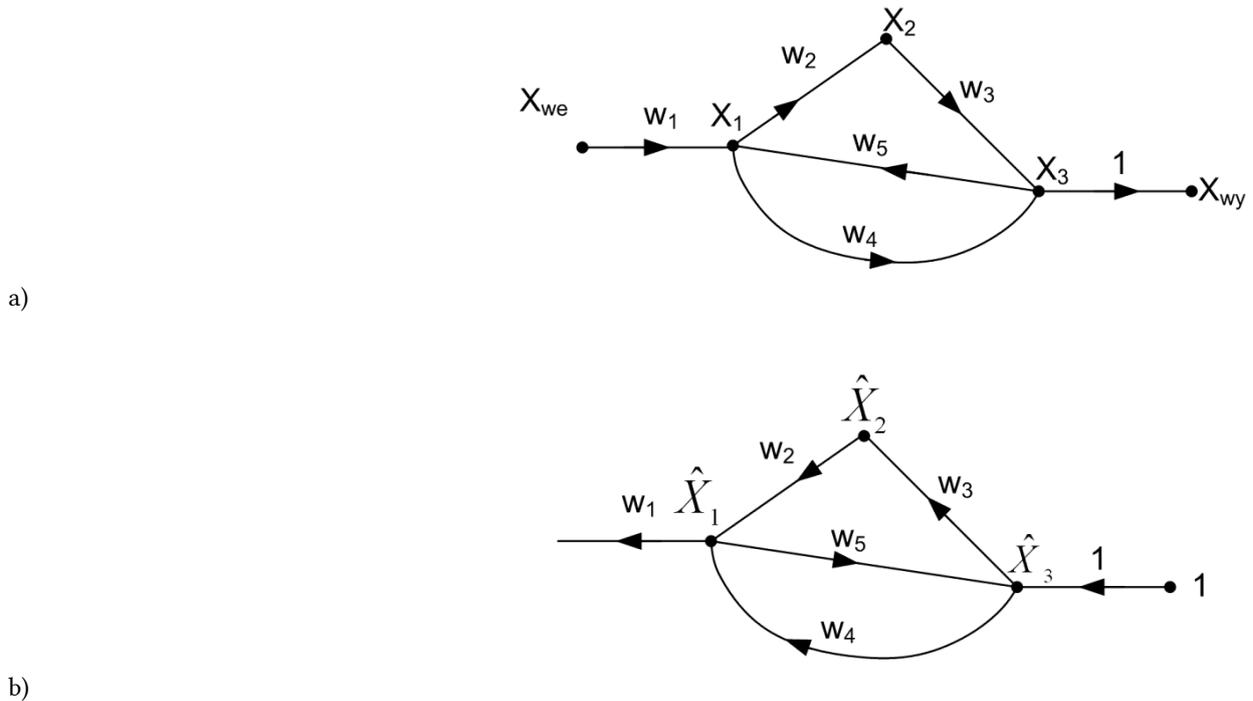


Fig. 4.8 The structure of: a) original graph G and b) adjoint graph \hat{G} in sensitivity computation.

Adjoint graph \hat{G} is shown in Fig. 4.8b. All signals in this graph are denoted with a hat. At such notations we get the following sensitivity formulas

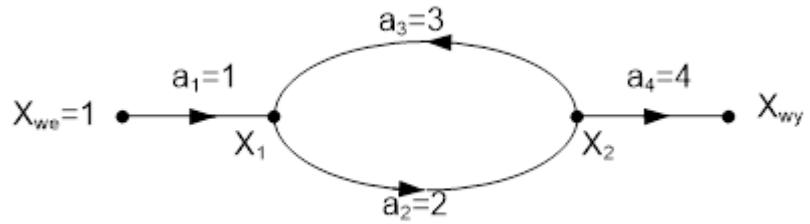
$$\begin{aligned}\frac{dX_{wy}}{dw_1} &= X_{we} \hat{X}_1 \\ \frac{dX_{wy}}{dw_2} &= X_1 \hat{X}_2 \\ \frac{dX_{wy}}{dw_3} &= X_2 \hat{X}_3 \\ \frac{dX_{wy}}{dw_4} &= X_1 \hat{X}_3 \\ \frac{dX_{wy}}{dw_5} &= X_3 \hat{X}_1\end{aligned}$$

Example 4.8

The next example will present the application of SFG in calculation of sensitivity of the transfer function $T=X_{wy}/X_{we}$, of the network represented by the original graph presented in Fig. 4.9a. Assume that all gains are known and equal: $a_1=1$, $a_2=2$, $a_3=3$, $a_4=4$. Assuming source signal $X_{we}=1$ the task of sensitivity of transfer function is transformed to the sensitivity of output node signal X_{wy} with respect to the gains of the branches.

The adjoint graph \hat{G} to the original system is presented in Fig. 4.9b.

a)



b)

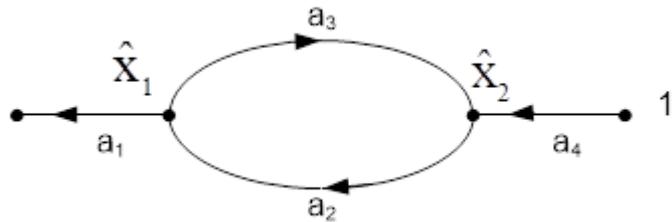


Fig. 4.9. Signal flow graph representing: a) original system, b) adjoint system.

From the Mason rule we get the solution of graph G as follows

$$X_1 = a_1 \frac{1}{1-a_2a_3} X_{we} = -\frac{1}{5}$$

$$X_2 = \frac{a_1a_2}{1-a_2a_3} X_{we} = -\frac{2}{5}$$

$$X_{wy} = \frac{a_1a_2a_4}{1-a_2a_3} X_{we} = -\frac{8}{5}$$

The same Mason rule applied to graph \hat{G} results in

$$\hat{X}_1 = \frac{a_2a_4}{1-a_2a_3} = -\frac{8}{5}$$

$$\hat{X}_2 = \frac{a_4}{1-a_2a_3} = -\frac{4}{5}$$

Hence the sensitivity values of transfer function T with respect to all gains are as follows

$$\frac{dT}{da_1} = 1 \cdot \hat{X}_1 = -\frac{8}{5}$$

$$\frac{dT}{da_2} = X_1 \hat{X}_2 = \frac{4}{25}$$

$$\frac{dT}{da_3} = X_2 \hat{X}_1 = \frac{16}{25}$$

$$\frac{dT}{da_4} = X_2 \cdot 1 = -\frac{2}{5}$$

The authenticity of the results can be checked by direct differentiation of the transfer function of the original graph obtained in an explicit way using Mason gain formula. This function is given by

$$T = \frac{a_1a_2a_4}{1-a_2a_3}$$

From the differentiation of this relations we get

$$\begin{aligned}\frac{dT}{da_1} &= \frac{a_2 a_4}{1 - a_2 a_3} = -\frac{8}{5} \\ \frac{dT}{da_2} &= \frac{a_1 a_4 (1 - a_2 a_3) + a_1 a_2 a_3 a_4}{(1 - a_2 a_3)^2} = \frac{4}{25} \\ \frac{dT}{da_3} &= \frac{a_1 a_2^2 a_4}{(1 - a_2 a_3)^2} = \frac{16}{25} \\ \frac{dT}{da_4} &= \frac{a_1 a_2}{(1 - a_2 a_3)^2} = -\frac{2}{5}\end{aligned}$$

As it is seen direct differentiation has confirmed the former results, following from the application of adjoint graph.

4.4. Incremental method of sensitivity computation

The incremental method of sensitivity computation requires also only two analysis. One should be done for original circuit and the second for the incremental one, supplied by the signal obtained from the original circuit analysis. The results of analysis of incremental circuit delivers immediately the proper sensitivity values.

The incremental method is complementary to the adjoint one [4,16]. At two analyses, the adjoint method allows calculating the sensitivity of one chosen output variable of the circuit (for example arbitrary voltage or current) to all internal parameters of the circuit. Contrary to this, the incremental method (applying also two analyses of the circuit) allows getting the sensitivity of all variables in the circuits (voltages and currents) to only one chosen internal parameter. Both methods operate with numerical values of elements and do not need the explicit form of solution of the circuit.

The fundamental point in this method is the relation describing change of output variable of the element at the change of input variable. Let us assume the passive element of the impedance Z . From the Ohm's law we have

$$U = ZI \quad (4.29)$$

The increment of the voltage U at small change ΔZ of the value of parameter Z is given by

$$\Delta U = \frac{\partial U}{\partial I} \Delta I + \frac{\partial U}{\partial Z} \Delta Z = Z \Delta I + I \Delta Z \quad (4.30)$$

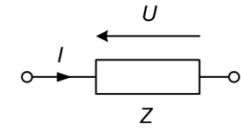
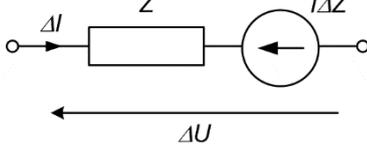
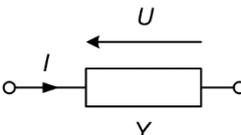
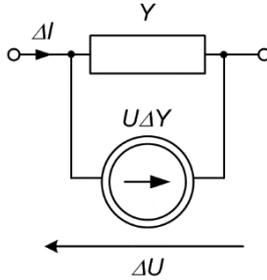
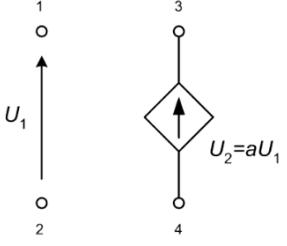
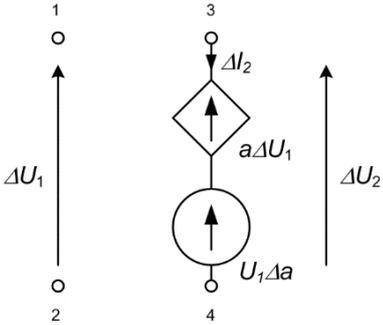
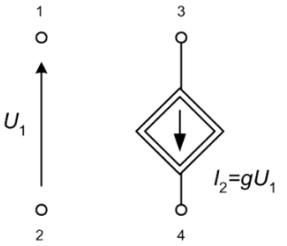
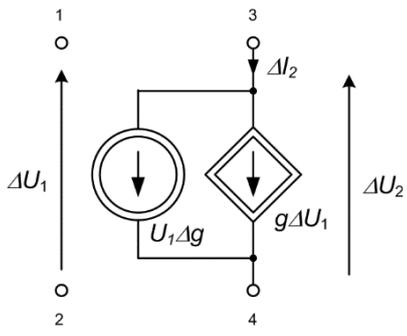
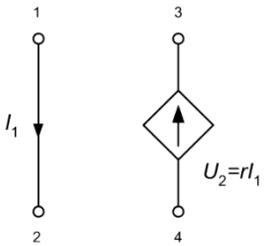
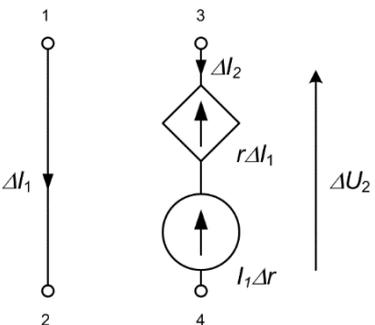
This equations represents the incremental model of element Z in which the current is represented by its increment ΔI and voltage by ΔU . It is composed of series connection of impedance Z and voltage source defined by $I \Delta Z$. The current I appearing in this equation is the value of current in the original circuit. The circuit model of it is depicted in the first row of table 4.4.

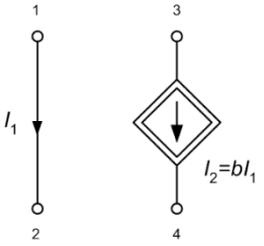
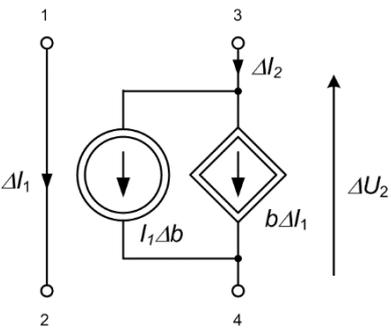
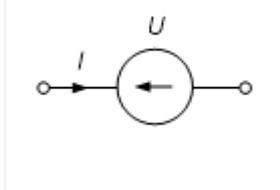
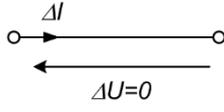
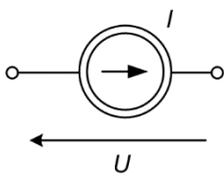
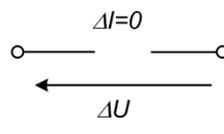
In an identical way we can create the incremental models of all other electrical elements, including controlled sources. As an example consider the voltage controlled voltage source VCVS described by $U_2 = aU_1$. The increment of the output voltage ΔU_2 is described by

$$\Delta U_2 = \frac{\partial U_2}{\partial U_1} \Delta U_1 + \frac{\partial U_2}{\partial a} \Delta a = a \Delta U_1 + U_1 \Delta a \quad (4.31)$$

This equations corresponds to the series connection of the incremental voltage controlled voltage source $U_1 \Delta a$ and the voltage source defined by $U_1 \Delta a$, where U_1 is the voltage value of the original circuit. The circuit model corresponding to this equation is depicted in the third row of table 4.4. The other rows in this table show the incremental models of other electrical elements forming the circuits. Observe, that each incremental model operates with a variable taken from the solution of the original circuit.

Table 4.4 Incremental models of the basic circuit elements.

Name of element	Original element	Incremental model of element
Impedance		
Admittance		
VCVS		
VCCS		
CCVS		

CCCS		
Independent voltage source		
Independent current source		

As it is seen from this presentation the incremental method requires 2 analyses of the circuit.

The first analysis is of original circuit and is directed to find the solution of voltage or current of the element with respect to which the sensitivity will be calculated.

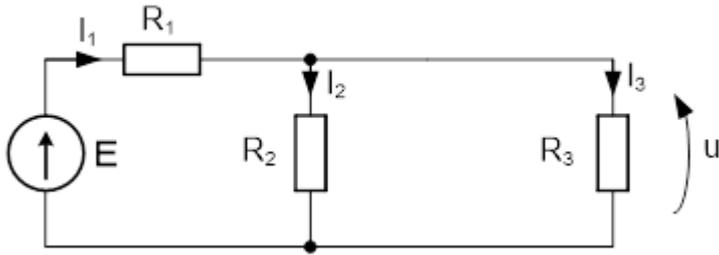
The second analysis is done for incremental model of the circuit. This model is built with emphasis to the element with respect to which the sensitivity is calculated. The other elements are the same as in original circuit. The sources from the original circuit are eliminated in the incremental model (as shown in the last two rows of table 4.4. The excitation for this model is the independent source associated with the incremented element.

The results of analysis of the incremental model deliver immediately the sensitivity of all variables in this circuit with respect to the particular incremented element, since all calculated variables represent increment of output signals ΔF_i as a function of increments of the element value Δx . If we assume $\Delta x=1$, the variables in incremental model represent the absolute sensitivity of F_i with respect to x .

Example 4.9

The incremental method will be illustrated on the example of circuit shown in Fig. 4.10a. Assume following values of elements: $E=40\text{V}$, $R_1=10\Omega$, $R_2=5\Omega$, $R_3=10\Omega$.

a)



b)

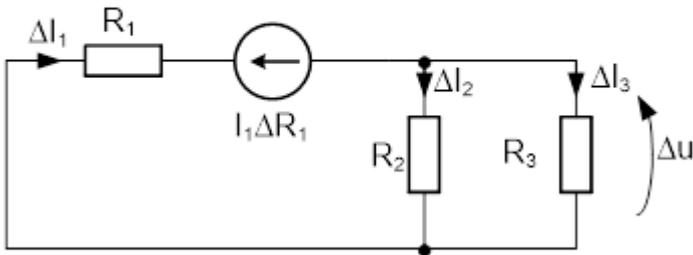


Fig. 4.10 The circuit under consideration in example: a) the original circuit, b) the incremental model with respect to changes of R_1 .

From the analysis of original circuit we get:

$$I_1 = \frac{E}{R_1 + \frac{R_2 R_3}{R_2 + R_3}} = 3$$

$$U = E \frac{R_2 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} = 10$$

$$I_2 = E \frac{R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} = 2$$

$$I_3 = E \frac{R_2}{R_1 R_2 + R_2 R_3 + R_1 R_3} = 1$$

Let us calculate the sensitivity of all currents in the circuit with respect to the resistance R_1 . The proper incremental model shown in Fig. 4.10b is built. From the analysis of this circuit we get the following results:

$$\Delta I_1 = - \frac{I_1 \Delta R_1}{R_1 + \frac{R_2 R_3}{R_2 + R_3}} = - \frac{9}{40} \Delta R_1$$

$$\Delta U = \Delta I_1 \frac{R_2 R_3}{R_2 + R_3} = - \frac{3}{4} \Delta R_1$$

$$\Delta I_2 = \frac{\Delta U}{R_2} = - \frac{3}{20} \Delta R_1$$

$$\Delta I_3 = \frac{\Delta U}{R_3} = - \frac{3}{40} \Delta R_1$$

Hence, the sensitivity values of all currents with respect to R_1 are equal:

$$\begin{aligned}\frac{\partial I_1}{\partial R_1} &= \frac{\Delta I_1}{\Delta R_1} = -\frac{9}{40} \\ \frac{\partial I_2}{\partial R_1} &= \frac{\Delta I_2}{\Delta R_1} = -\frac{3}{20} \\ \frac{\partial I_3}{\partial R_1} &= \frac{\Delta I_3}{\Delta R_1} = -\frac{3}{40}\end{aligned}$$

Observe, that the value of ΔR_1 did not matter in final calculations. Therefore, we can assume $\Delta R_1=1$ without any change of the final sensitivity values.

To check the correctness of the results we can apply direct differentiation of the symbolic formulas describing I_1 , I_2 and I_3 with respect to R_1 . As a result we get

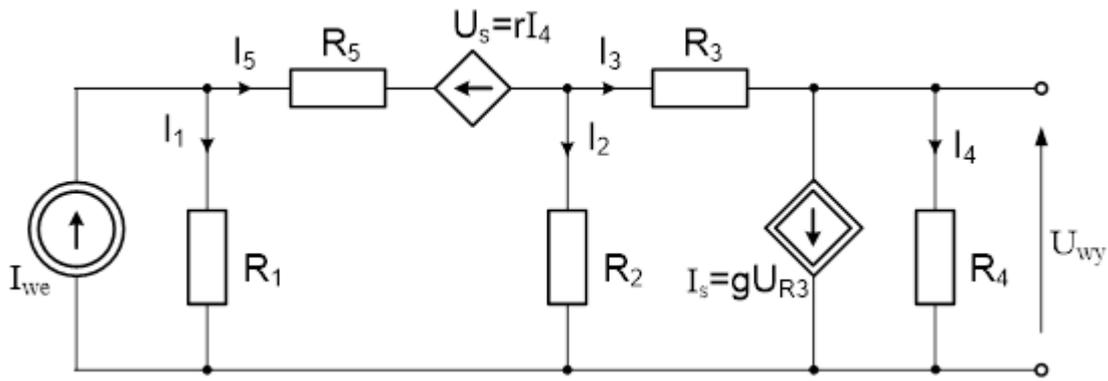
$$\begin{aligned}\frac{\partial I_1}{\partial R_1} &= E \frac{-(R_2+R_3)^2}{(R_1 R_2+R_2 R_3+R_1 R_3)^2} = -\frac{9}{40} \\ \frac{\partial I_2}{\partial R_1} &= E \frac{-R_3(R_2+R_3)^2}{(R_1 R_2+R_2 R_3+R_1 R_3)^2} = -\frac{3}{20} \\ \frac{\partial I_3}{\partial R_1} &= E \frac{-R_2(R_2+R_3)^2}{(R_1 R_2+R_2 R_3+R_1 R_3)^2} = -\frac{3}{40}\end{aligned}$$

It is evident that the corresponding results are the same. The important advantage of incremental method is that all calculations are done in numerical way and we don't need to have the symbolic solution for the variables of circuit.

Example 4.10

As the second example let us consider the more complicated circuit containing 2 controlled sources presented in Fig. 4.11a. The sensitivity will be calculated with respect to the coefficient r of the current controlled voltage source $U_s=rI_4$. The corresponding incremental model is shown in Fig. 4.11b. Observe that all variables in this circuit represent increments of the voltages and currents.

a)



b)

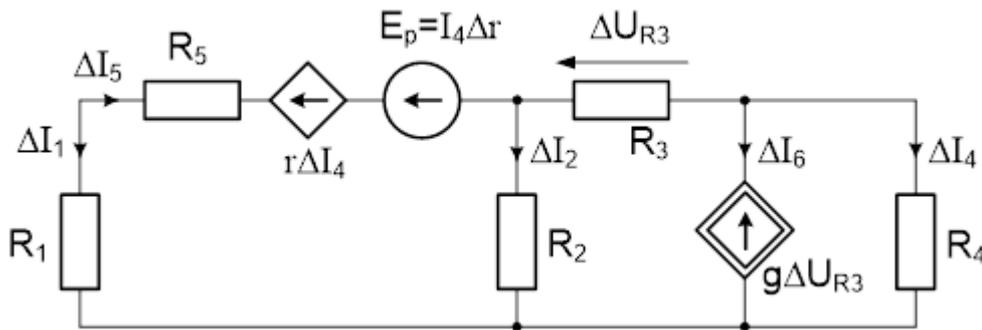


Fig. 4.11 The circuit under consideration in example: a) the original circuit, b) the incremental model with respect to changes of the controlling coefficient r .

This time the analyses of both circuits are done using Matlab program, shown below.

```
%Sensitivity analysis using incremental method
%Elements
G1=1; G2=4; G3=2; G4=2.5; G5=2; r=2; g=3;
Iwe=1;

%Analysis of original circuit – nodal method
-G5 G5+G2+G3 -G3;
0 -G3+g G3+G4-g];
Iz=[Iwe;0;0];
Vo=Y\Iz;
Uwy=Vo(3);

%Notations: UG1=UR1, UG2=UR2, UG3=UR3, UG4=UR4, UG5=UR5, Ur=Us, Ig=Is,
UG1=Vo(1); UG5=Vo(1) -Vo(2) - r*G4*Vo(3); UG2=Vo(2); UG3=Vo(2) -Vo(3);
UG4=Vo(3); Ior=Vo(3)*G4; Uog=UG3; Uor=r*G4*Vo(3);

%Analysis of incremental circuit with respect to changes of r
%
dr=1;
```

```

Yp=Y;
Izp=[Ior*dr*G5; -Ior*dr*G5; 0];
Vp=Yp\Izp;
Upwy=Vp(3);
%Notations: UpG1=ΔUR1, UpG2=ΔUR2, UpG3=ΔUR3, UpG4=ΔUR4, UpG5=ΔUR5, Upr=ΔUs, Ipg=ΔI
UpG1=Vp(1); UpG2=Vp(2); UpG3=Vp(2)-Vp(3); UpG5=Vp(1)-Vp(2)-r*G4*Vp(3)-dr*Ior;
Upr=r*G4*Vp(3)+dr*G4*Vo(3); UpG4=Vp(3); Ipg=g*UpG3;

% Absolute sensitivity
BUG1= UpG1/dr; BUG2=UpG2/dr; BUG3=UpG3/dr; BUG4=UpG4/dr; BUG5=UpG5/dr; BIg=Ipg/dr;
disp('Absolute sensitivity');
BUG=[BUG1 BUG2 BUG3 BUG4 BUG5 BIg BUr]

%relative sensitivity
SUG1=BUG1*G1/UG1; SUG2=BUG2*G2/UG2; SUG3=BUG3*G3/UG3; SUG4=BUG4*G4/UG4;
SUG5=BUG5*G5/UG5; SIg=BIg*g/Uog; SUR=BUr*r/Uor;
disp('Relative sensitivity');
SU=[SUG1 SUG2 SUG3 SUG4 SUG5 SIg SUR]

```

As a result of such analyses we get:

The absolute sensitivity of all voltages and controlled current with respect to r

$$\left[\frac{\partial U_{R1}}{\partial r} \quad \frac{\partial U_{R2}}{\partial r} \quad \frac{\partial U_{R3}}{\partial r} \quad \frac{\partial U_{41}}{\partial r} \quad \frac{\partial U_{R5}}{\partial r} \quad \frac{\partial I_s}{\partial r} \quad \frac{\partial U_s}{\partial r} \right] =$$

$$[-0.0670 \quad 0.0048 \quad 0.0080 \quad -0.0032 \quad 0.0335 \quad 0.0239 \quad -0.1052]$$

The relative sensitivity values

$$\left[S_r^{U_{R1}} \quad S_r^{U_{R2}} \quad S_r^{U_{R3}} \quad S_r^{U_4} \quad S_r^{U_5} \quad S_r^{I_s} \quad S_r^{U_s} \right] =$$

$$[-0.2679 \quad 0.3571 \quad 0.1786 \quad 0.2232 \quad 0.1786 \quad 0.8036 \quad 1.1786]$$

4.5. Example of sensitivity analysis in complex frequency

Both sensitivity analyses presented above are applicable in both purely resistive and RLC circuit at sinusoidal excitation. In this point we will demonstrate the procedure on the example of integrating RC circuit built on the basis of operational amplifier of the amplification gain A as shown in Fig. 4.12.

We will investigate the relative sensitivity of output voltage U_{wy} with respect to resistance R , admittance Y_C (capacitance C) of capacitor and gain A of operational by using adjoint network method. Fig. 4.13a presents the equivalent circuit of operational amplifier and Fig. 4.13b the corresponding adjoint network, where $U_{wy} = -AU$.

a)

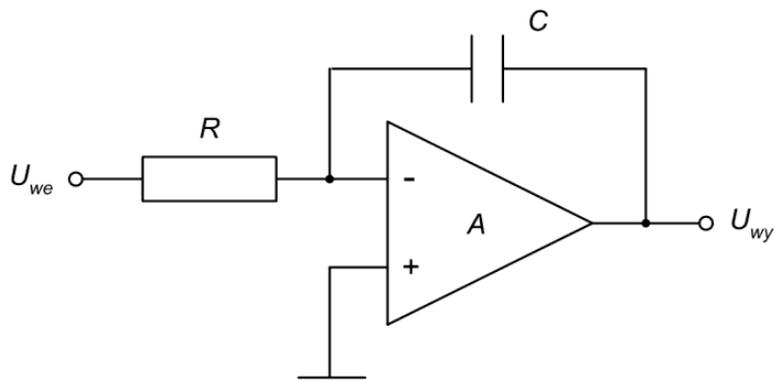


Fig. 4.12. RC integrator built on the basis of operational amplifier of finite gain.

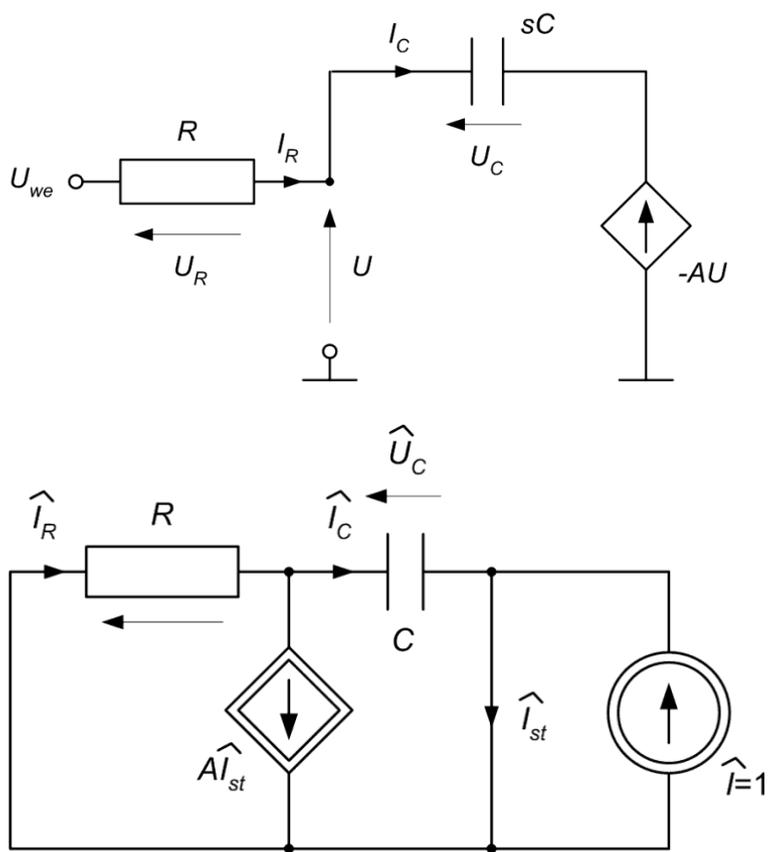


Fig. 4.13. The equivalent circuit of integrator: a) the original structure, b) the adjoint circuit.

According to the table 4.3 the absolute sensitivity functions are described by

$$B_R^{U_{wy}} = I_R \hat{I}_R \quad (4.32)$$

$$B_{Y_C}^{U_{wy}} = -U_C \hat{U}_C \quad (4.33)$$

$$B_C^{U_{wy}} = j\omega B_{Y_C}^{U_{wy}} \quad (4.34)$$

$$B_K^{U_{wy}} = U \hat{I}_{st} \quad (4.35)$$

Therefore both analyses should deliver the currents of resistor, voltages of capacitor and controlling variables

of the VCVS in original and adjoint network. On the basis of Kirchhoff's laws the original circuit of Fig. 4.11a is described by the following equations

$$\begin{bmatrix} U_{we} \\ 0 \\ U_{we} \end{bmatrix} = \begin{bmatrix} R & 1 & -A \\ 1 & -Y_C & 0 \\ R & 0 & 1 \end{bmatrix} \begin{bmatrix} I_R \\ U_C \\ U \end{bmatrix}$$

In a similar way we describe the adjoint circuit

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -Y_C & -A \\ R & 1 & 0 \\ 0 & -Y_C & 1 \end{bmatrix} \begin{bmatrix} \hat{I}_R \\ \hat{U}_C \\ \hat{I}_{st} \end{bmatrix}$$

Both equations have been coded in Matlab program. The numerical results of computations have been used to find absolute and relative sensitivity to the resistance, admittance of capacitor, capacitance and the gain of operational amplifier (this gain was assumed to be constant $A=K$ and independent on frequency). The sensitivity functions depend on frequency in RC circuits.

Additionally, the numerical results of application of adjoint method have been compared to the theoretical values obtained by direct differentiation of the output voltage described in an explicit way, where $U_{wy} = U_{we} * A / (1 + Y_C * R(1 + A))$. The results of both approaches to sensitivity computation agree completely.

```
% Sensitivity of RC integrator
```

```
format compact
```

```
omega=[10:5:2000];
```

```
L=length(omega);
```

```
R=1000; C=1e-6; A=5; Uwe=1;
```

```
K=A;
```

```
% Analysis of original circuit
```

```
% Vector x=[IR; UC; U]'
```

```
for i=1:L
```

```
YC=j*omega(i)*C;
```

```
A1=[R 1 -K
```

```
1 -YC 0
```

```
R 0 1];
```

```
b=[Uwe; 0; Uwe];
```

```
x=A1\b;
```

```
Uwy=-K*x(3);
```

```
% Analysis of adjoint circuit
```

```
% xd=[IRd UCd Ist]'
```

```
Ad=[1 -YC -K
```

```
R 1 0
```

```
0 -YC 1];
```

```
bd=[0; 0 ; 1];
```

```

xd=Ad\bd;

% Absolute sensitivity
BUR=x(1)*xd(1);
BUYC=-x(2)*xd(2);
BUC=BUYC*j*omega(i);
BUK=x(3)*xd(3);

% Realtive sensitivity
SUR(i)=BUR*R/Uwy;
SUC(i)=BUC*C/Uwy;
SUYC(i)=BUYC*YC/Uwy;
SUK(i)=BUK*K/Uwy;

% Analytical differentiation
% Uwy = Uwe*A/(1+YC*R(1+A))
BURa=Uwe*K*(YC+K*YC)/(1+YC*R+K*YC*R)^2;
BUYCa=Uwe*K*(R+K*R)/(1+YC*R+K*YC*R)^2;
BUCa=Uwe*BUYCa*j*omega(i);
BUKa=-Uwe*(1+YC*R)/(1+YC*R+K*YC*R)^2;
end

% Graphical results
plot(omega, abs(SUR), '- ', omega, abs(SUYC), '- - ', omega, abs(SUK), '- . ');grid
xlabel('omega');ylabel('Relative sensitivity')
legend('Resistor', 'Capacitor', 'Gain')

```

Fig. 4.14 shows the absolute values of sensitivity of output signal with respect to the resistance R , capacitance C and the gain A as a function of frequency [16].

4.6 Project devoted to sensitivity analysis

One of the project, student should do, is to prepare the Matlab program for calculating the sensitivity of the circuit (particular voltage or current of the branch indicated by the teacher) to all internal parameters of the circuit. This can be easily done using the adjoint method. An example of the circuit subject to sensitivity analysis of the output voltage is presented below.

The circuit is described by the nodal equations. The values of elements can be assumed in an arbitrary way. The lines of Matlab program for sensitivity computation are shown below.

```

% Sensitivity project
r1=1; r2=2, r3=3, r4=4, r5=5, r6=6, alfa=2.5, ii=1

```

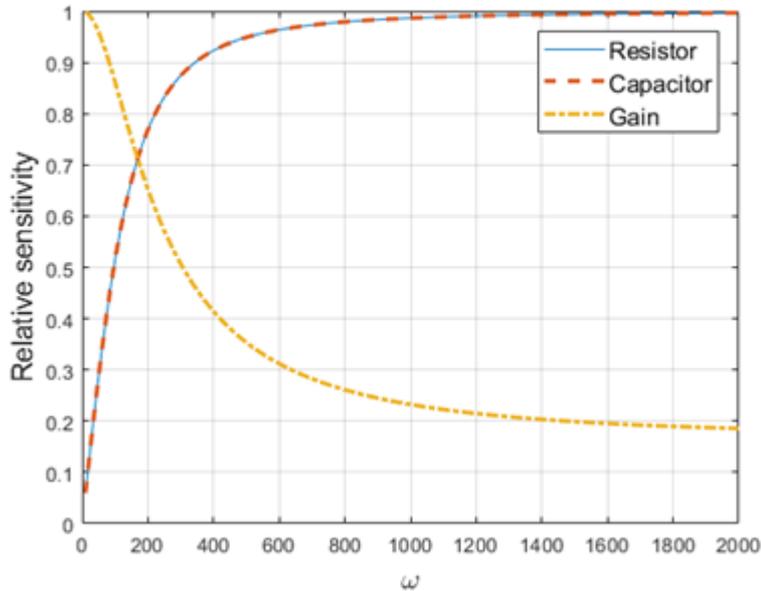


Fig. 4.14. Sensitivity characteristics of the circuit versus frequency.

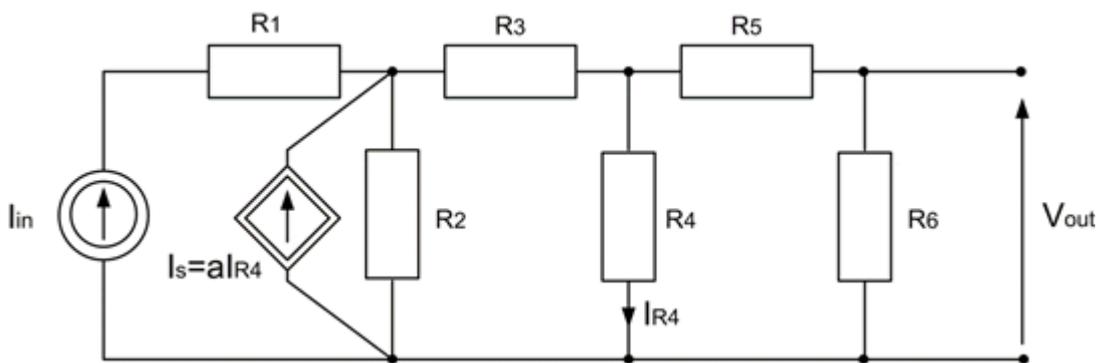


Fig. 4.15 The circuit structure subject to sensitivity analysis.

$$g_1=1/r_1; \quad g_2=1/r_2; \quad g_3=1/r_3; \quad g_4=1/r_4; \quad g_5=1/r_5; \quad g_6=1/r_6;$$

%Original network

$$y = \begin{bmatrix} g_1 & -g_1 & 0 & 0 \\ -g_1 & g_1+g_2+g_3 & -g_3-\alpha g_4 & 0 \\ 0 & -g_3 & g_3+g_4+g_5 & -g_5 \\ 0 & 0 & -g_5 & g_5+g_6 \end{bmatrix}$$

% Branch currents of original circuit

$$i_s = [i_i \quad 0 \quad 0 \quad 0]'$$

$$v = \text{inv}(y) * i_s$$

$$i_{r1} = i_i; \quad i_{r2} = g_2 * v(2); \quad i_{r3} = g_3 * (v(2) - v(3));$$

$$i_{r4} = g_4 * v(3); \quad i_{r5} = g_5 * (v(3) - v(4)); \quad i_{r6} = g_6 * v(4);$$

$$v_{out} = v(4);$$

```

%Adjoint network
ya=y'
isa=[0 0 0 1]'
va=inv(ya)*isa

% Currents and voltages in adjoint network
iar1=0; iar2=g2*va(2); iar3=g3*(va(2)-va(3));
iar4=g4*(va(3)+alfa*va(2)); iar5=g5*(va(3)-va(4));
iar6=g6*va(4); var2=-va(2);

%Absolute sensitivity
br1=ir1*iar1;
br2=ir2*iar2;
br3=ir3*iar3;
br4=ir4*iar4;
br5=ir5*iar5;
br6=ir6*iar6;
balfa=-ir4*var2;

%Relative sensitivity
sr1=br1*r1/vout;
sr2=br2*r2/vout;
sr3=br3*r3/vout;
sr4=br4*r4/vout;
sr5=br5*r5/vout;
sr6=br6*r6/vout;
salfa=balfa*alfa/vout;
s=[sr1 sr2 sr3 sr4 sr5 sr6 salfa]

```

Student should prepare the similar program for his own circuit structure, calculate the absolute and relative sensitivity values and check the validity of the appropriate invariance criteria.

4.7 Exercises

Calculate the relative sensitivity S_x^F of the function $F(x)$ with respect to variable x for $x=1$.

$$F(x) = \frac{\sqrt{2x^2 + x}}{3x^3 + 2x + 1}$$

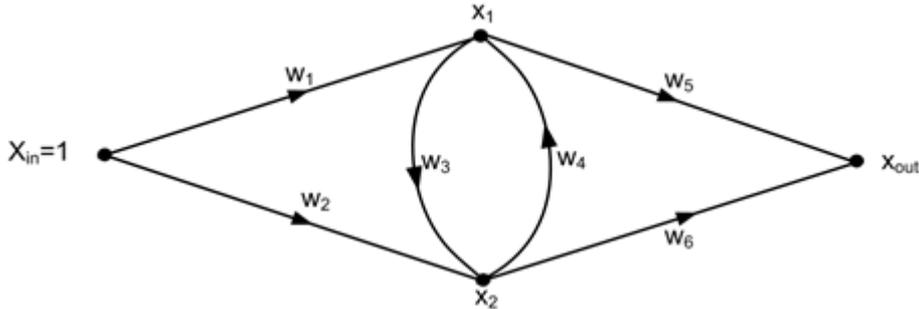
Solution:

Let us present $F(x)$ in the form $F(x)=F_1(x)/F_2(x)$, where $F_1(x)$ represents the numerator and $F_2(x)$ denominator. In such case we have

$$S_x^F(x) = S_x^{F_1}(x) - S_x^{F_2}(x) = 0.5(2x+1) \frac{x}{2x^2+x} - \frac{(9x^2+2)x}{3x^3+2x+1} = -4/3$$

Apply adjoint graph to calculate the relative sensitivity of x_{out} with respect to all weights of the graph. Assume $w_1=1, w_2=0.5, w_3=1, w_4=2, w_5=-2, w_6=3$.

a)



b)

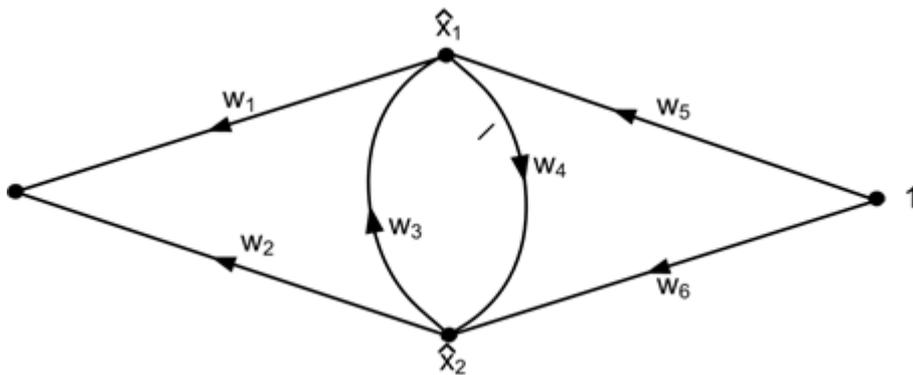


Fig.4.16 Graph of original system (a) and of adjoint system (b)

Solution:

Applying Mason formula for original SFG we get

$$x_1 = \frac{w_1 + w_2 w_4}{1 - w_3 w_4} = 0.4$$

$$x_2 = \frac{w_2 + w_1 w_3}{1 - w_3 w_4} = -0.3$$

$$x_{out} = \frac{w_2 w_4 w_5 + w_1 w_3 w_6}{1 - w_3 w_4} = -1.6$$

On the basis of Mason formula applied for adjoint graph we get

$$\hat{x}_1 = \frac{w_5 + w_6 w_3}{1 - w_3 w_4} = -1.6$$

$$\hat{x}_2 = \frac{w_6 + w_4 w_5}{1 - w_3 w_4} = -0.2$$

The sensitivity functions are then as follows

$$\begin{aligned} \frac{\partial x_{out}}{\partial w_1} &= 1 \cdot \hat{x}_1 = -1.6 \rightarrow S_{w_1}^{x_{out}} = -1.6 \frac{1}{-1.6} = 1 \\ \frac{\partial x_{out}}{\partial w_2} &= 1 \cdot \hat{x}_2 = -0.2 \rightarrow S_{w_2}^{x_{out}} = -0.2 \frac{0.5}{-1.6} = 0.0625 \\ \frac{\partial x_{out}}{\partial w_3} &= x_1 \cdot \hat{x}_2 = -0.08 \rightarrow S_{w_3}^{x_{out}} = -0.08 \frac{-2}{-1.6} = 0.1 \\ \frac{\partial x_{out}}{\partial w_4} &= x_2 \cdot \hat{x}_1 = 0.48 \rightarrow S_{w_4}^{x_{out}} = 0.48 \frac{2}{-1.6} = -0.6 \\ \frac{\partial x_{out}}{\partial w_5} &= x_1 \cdot 1 = 0.4 \rightarrow S_{w_5}^{x_{out}} = 0.4 \frac{-2}{-1.6} = -0.5 \\ \frac{\partial x_{out}}{\partial w_6} &= x_2 \cdot 1 = -0.3 \rightarrow S_{w_6}^{x_{out}} = -0.3 \frac{3}{-1.6} = -0.5625 \end{aligned}$$

Build adjoint network for the circuit presented below

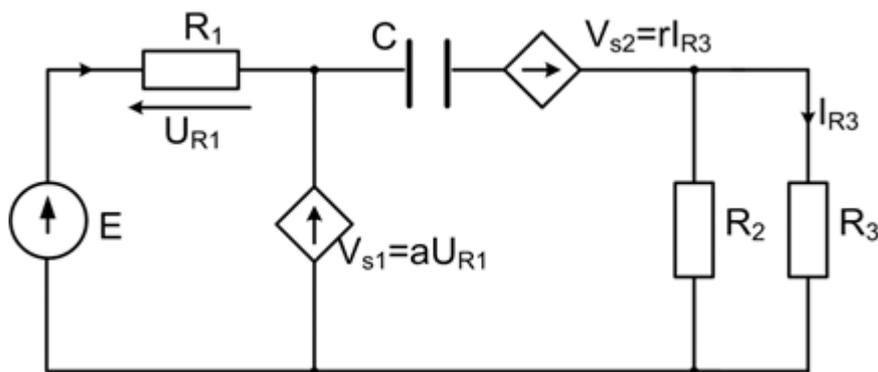


Fig. 4.17a Original circuit structure

Solution:

The network structure adjoint to the original circuit is presented below.

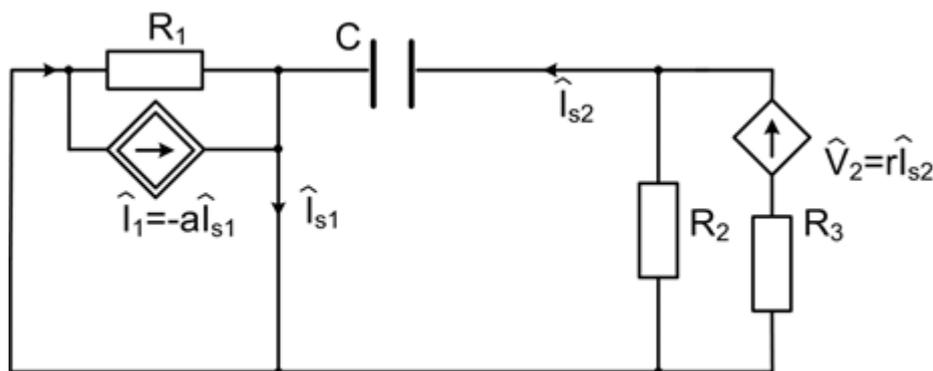


Fig. 4.17b Circuit structure adjoint to the original circuit

Calculate sensitivity of all currents in the circuit of Fig. 4.18a with respect to the gain $a=2$ of the current controlled current source (CCCS).

Solution:

This time we apply the incremental method of sensitivity computation. The incremental circuit is presented in Fig. 4.3b. In the first phase we analyse the original circuit. From Kirkhoff's law we get the following system of equations

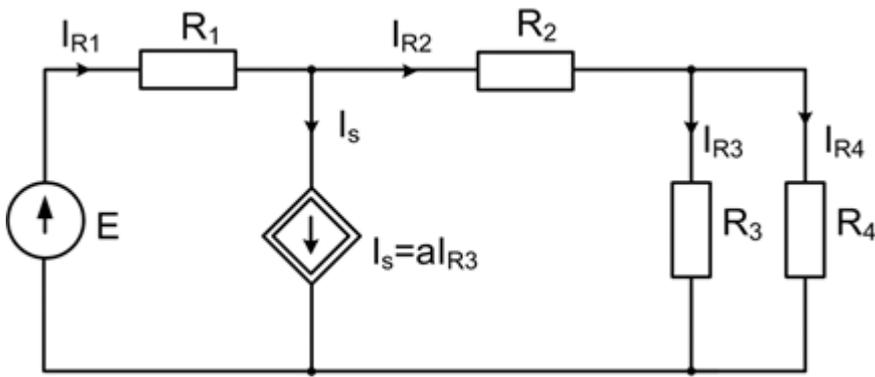


Fig. 4.18a Original circuit structure subject to sensitivity analysis

$$I_{R3} = I_{R4}$$

$$40 = 5I_{R1} + 5I_{R2} + 10I_{R3}$$

$$I_{R1} = I_{R2} + aI_{R3}$$

$$I_{R2} = 2I_{R3}$$

The solution of these equations delivers the results:

$$I_{R1} = 4, I_{R2} = 2, I_{R3} = I_{R4} = 1, I_s = 2$$

The incremental circuit structure is presented in Fig. 4.3b.

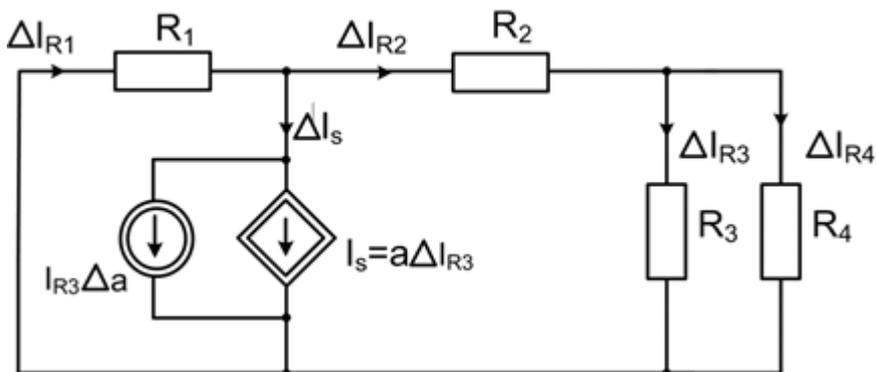


Fig. 4.18b The incremental circuit structure corresponding to the original circuit of Fig. 4.3a

Kirchhoff's set of equations for incremental circuit is as following

$$\Delta I_{R3} = \Delta I_{R4}$$

$$\Delta I_{R1} = a\Delta I_{R3} + \Delta I_{R2} + I_{R3}\Delta a$$

$$\Delta I_{R2} = 2\Delta I_{R3}$$

$$5\Delta I_{R1} + 5\Delta I_{R2} + 10\Delta I_{R3} = 0$$

Solving these equations we get

$$\Delta I_{R1} = \frac{\Delta a}{2}, \Delta I_{R2} = -\frac{\Delta a}{4}, \Delta I_{R3} = -\frac{\Delta a}{8}, \Delta I_{R4} = -\frac{\Delta a}{8}, \Delta I_s = \frac{3\Delta a}{4}$$

Hence the sensitivity functions are as follows

$$\frac{\partial I_{R1}}{\partial a} = \frac{1}{2} \rightarrow S_a^{I_{R1}} = \frac{1}{2} \frac{a}{I_{R1}} = 1/4$$

$$\frac{\partial I_{R2}}{\partial a} = -\frac{1}{4} \rightarrow S_a^{I_{R2}} = -\frac{1}{4} \frac{a}{I_{R2}} = -1/4$$

$$\frac{\partial I_{R3}}{\partial a} = -\frac{1}{8} \rightarrow S_a^{I_{R3}} = -\frac{1}{8} \frac{a}{I_{R3}} = -1/4$$

$$\frac{\partial I_{R4}}{\partial a} = -\frac{1}{8} \rightarrow S_a^{I_{R4}} = \frac{1}{8} \frac{a}{I_{R4}} = -1/4$$

$$\frac{\partial I_s}{\partial a} = \frac{3}{4} \rightarrow S_a^{I_s} = \frac{3}{4} \frac{a}{I_s} = 3/4$$

4.8 Basic definitions

Adjoint circuit – the circuit is adjoint to the original circuit, if it has the same topology, and hybrid descriptions of both circuit satisfy special conditions.

Adjoint graph – the signal flow graph which has the same topology as the original one, but inverted direction of signal flow.

Adjoint method – the method of determining the derivative of network variable (current or voltage) with respect to all element parameters. It produces exact result of derivative without knowing the explicit form of solution for the considered variable. It needs only two analyses: one of the original circuit and the other of adjoint one.

Controlled sources – the current and voltage sources of the values dependent on the other branch voltages or current in the circuit. They form in general active devices.

Hybrid description – matrix description of linear multiport network arranging the voltage and currents of the terminal in a very specified form.

Incremental circuit – the circuit in which all elements have been replaced by their increments, including the variables (currents and voltages).

Incremental method – the method of determining the derivative of all network variables (currents and voltages) with respect to one chosen element parameter. It produces exact result of derivative without knowing the explicit form of solution for the considered variables. Only two analyses are needed: one of the original circuit and the other of incremental one.

Relative sensitivity – the normalized value of the sensitivity. It is interpreted as the ratio of the fractional change of the network function F to the fractional change in the parameter x , provided that all changes are sufficiently small.

Sensitivity – the relationship indicating what is the ratio between the small change in electrical output signal F to a small change in physical input parameter x . It is defined in mathematical term as the derivative of the output signal with respect to the input parameter.

Sensitivity invariants – the property of certain sensitivity functions which are preserved under some transformation of circuit parameters.

Tellegen theorem – the most powerful theorem in [network theory](#), stating that sum of products of currents and voltages over all elements in the circuit is zero. It is valid for any network that contains any elements, linear or [nonlinear](#), passive or active, time-varying or time-invariant. Moreover, the currents and voltages may come from an entirely different networks, so long as these two networks have the same topology.

5. Discrete time systems

The circuits considered up to now belonged to the continuous time systems (called also analog systems), which are described in the continuous time or in the complex frequency domain s at application of Laplace transformation. Their general dynamic description in time domain is usually assumed in the state space form [6,7,16]

$$\begin{aligned}\frac{dx}{dt} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}\quad (5.1)$$

in which \mathbf{A} , \mathbf{B} , \mathbf{C} , \mathbf{D} are the state matrices, \mathbf{x} – state vector, \mathbf{u} – excitation vector, \mathbf{y} – output vector. Solution of this system of differential equations in time domain can be easily obtained using numerical methods, for example implemented in Simulink [17,21]. In the complex frequency domain the continuous time description is usually characterized by the transfer function matrix $\mathbf{H}(s)$, which is related to state space matrices as follows

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{1} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\quad (5.2)$$

This transfer function for linear lumped parameter systems takes the form of rational function. For one input and one output system it can be expressed in the scalar form

$$H(s) = \frac{L(s)}{M(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}\quad (5.3)$$

Discrete systems are contrasted with continuous systems, because they are represented in discrete time. In this type of system the information of signals is accessible only at discrete time points $tk=kTs$, where Ts is a sampling period and k the time index. Because discrete systems have a countable number of states, they may be described in precise [mathematical models](#). This chapter will be devoted to different aspects of such discrete time systems.

5.1. Description of discrete system in time domain

The discrete systems may be created in different ways. The typical one is measuring the signals in discrete time points following from sampling clock ticks. The general description of discrete systems may be presented in the form of state space, analogously as it was done for continuous systems [16,18]

$$\begin{aligned}\mathbf{x}(n+1) &= \mathbf{A}\mathbf{x}(n) + \mathbf{B}\mathbf{u}(n) \\ \mathbf{y}(n) &= \mathbf{C}\mathbf{x}(n) + \mathbf{D}\mathbf{u}(n)\end{aligned}\quad (5.4)$$

This description uses discrete time notation n , in which the samples are described for discrete time $tn=nTs$. In the case of one input and one output system of the signals $u(n)$ for input and $y(n)$ of output the discrete system is often presented in a simplified form of difference equation

$$y(n) = b_0 u(n) + b_1 u(n - 1) + \dots + b_M u(n - M) + a_1 y(n - 1) + \dots + a_N y(n - N) \tag{5.5}$$

It should be noted, that discrete system described by general equation (5.4) is stable, when all eigenvalues of the matrix A are inside of the unit circle (the circle with a radius of one).

It is possible to change from continuous to discrete time description by substituting the differentiation by finite differences of the variables. There are many such transformation formulas. The simplest one is direct application of definition of derivative, by assuming finite value of variable (usually time) with respect to which the derivative is calculated. Assuming finite increase of variables we can obtain two different formulas of approximation. Let us assume the variable x under differentiation at the time point $tn=nTs$, that is $x(tn)=x(n)$ and the finite increase of time $rt=Ts$. In such case we define [6,16]

- forward Euler formula (extrapolation)

$$\frac{dx}{dt} = \frac{x(n + 1) - x(n)}{T_s} \tag{5.6}$$

- backward Euler formula (interpolation)

$$\frac{dx}{dt} = \frac{x(n) - x(n - 1)}{T_s} \tag{5.7}$$

They represent one-step integration formulas. More advanced are multi-step Gear formulas, which approximate derivative by taking into account many time points from the past. Table 5.1 shows the second, third and fourth order Gear formulas as the approximation of the derivative of variable x in the time point $tn=nTs$

Table 5.1. Gear approximation formulas of derivative.

Order of approximation	Gear formula
2	$\frac{dx}{dt} = \frac{3}{2T_s} \left[x(n) - \frac{4}{3}x(n - 1) + \frac{1}{3}x(n - 2) \right]$
3	$\frac{dx}{dt} = \frac{11}{6T_s} \left[x(n) - \frac{18}{11}x(n - 1) + \frac{9}{11}x(n - 2) - \frac{2}{11}x(n - 3) \right]$
4	$\frac{dx}{dt} = \frac{25}{12T_s} \left[x(n) - \frac{48}{25}x(n - 1) + \frac{36}{25}x(n - 2) - \frac{16}{25}x(n - 3) + \frac{3}{25}x(n - 4) \right]$

Application of approximation of derivative by finite differences changes the differential equation into difference one. Number of points from the past is dependent on the applied algorithm. The accuracy of approximation is dependent on the type of applied algorithm, order of approximation (in Gear method) and the length of sampling period.

Observe, that substituting the differential equations by the difference one simplifies the process of solution of such systems, since the difficult process of integration is transformed to simple adding terms from the past. The latter operations can be easily coded into the form of computer program.

Example 5.1

These aspects of approximation will be illustrated on the example of differential equation $\frac{dx}{dt} = 3t^2$ by applying both Euler methods and Gear method of the second and third orders by assuming $T_s=0.5$. The numerical results for some chosen values of t are presented in Table 5.2. The second column shows the exact values and other columns the approximations of different types.

Table 5.2. The approximation results of $\frac{dx}{dt} = 3t^2$ at application of Euler and Gear methods.

tn	$\frac{dx}{dt} = 3t_n^2$	Forward Euler	Backward Euler	2-nd order Gear	3-rd order Gear
1.50	6.75	9.25	4.75	6.25	6.75
2.00	12.00	15.25	9.25	11.50	12.00
2.50	18.75	22.75	15.25	18.25	18.75
3.00	27.00	31.75	22.75	26.50	27.00
3.50	36.75	42.25	31.75	36.25	36.75
4.00	48.00	54.25	42.25	47.50	48.00
4.50	60.75	67.75	54.25	60.25	60.75
5.00	75.00	82.75	67.75	74.50	75.00
5.50	90.75	99.25	82.75	90.25	90.75
6.00	108.00	117.25	99.25	107.50	108.00

As it is seen, the most accurate results correspond to application of 3-rd order Gear method. Both Euler methods delivered results far from expected, the forward one overestimated and backward underestimated. Fig. 5.1 presents the changes of estimation errors as a function of time. The Gear methods deliver not only the smallest errors, but also stable with time, which is not true for Euler methods.

Example 5.2

As an example consider the analog RLC circuit structure presented in Fig. 5.2. We will show how to transform it into the discrete system by applying Euler rules.

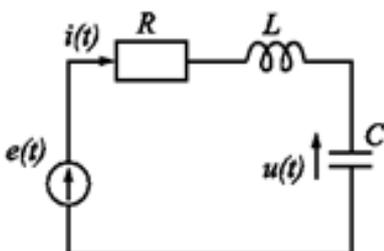


Fig. 5.2. The RLC circuit structure in the example.

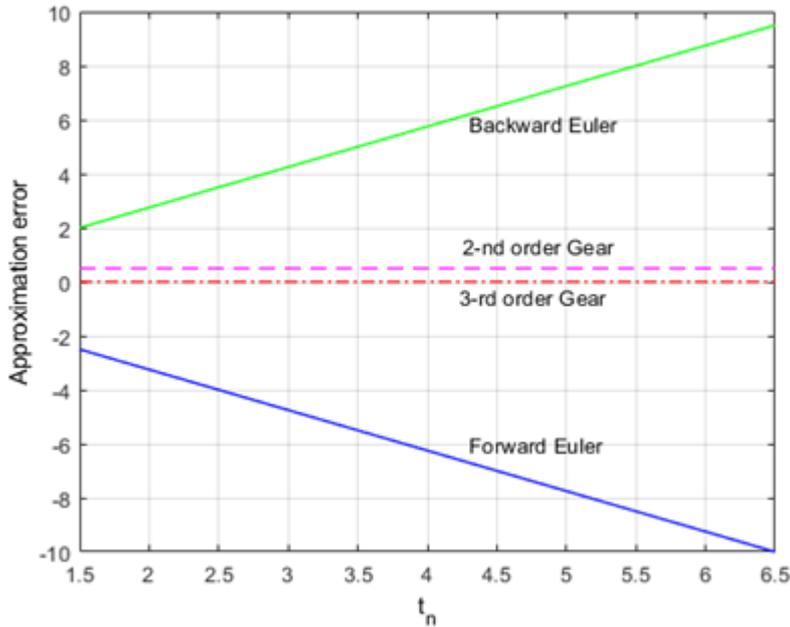


Fig. 5.1. Approximation errors versus time for different methods of approximation.

From the Kirchhoff's equations we can define the normal form of state space description of the circuit

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{du}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i \\ u \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} e(t)$$

where i is the current of inductor and u voltage of the capacitor. The excitation is the voltage source $e(t)$. Application of the forward Euler formula for $tn=nTs$ changes this differential equations into the difference one described in the explicit matrix form as follows

$$\begin{bmatrix} i(n+1) \\ u(n+1) \end{bmatrix} = \begin{bmatrix} 1 - \frac{RT_s}{L} & -\frac{T_s}{L} \\ \frac{T_s}{C} & 1 \end{bmatrix} \begin{bmatrix} i(n) \\ u(n) \end{bmatrix} + \begin{bmatrix} \frac{T_s}{L} \\ 0 \end{bmatrix} e(n)$$

On the other side, applying the backward Euler formula leads to the implicit form of matrix description

$$\begin{bmatrix} 1 + \frac{RT_s}{L} & \frac{T_s}{L} \\ -\frac{T_s}{C} & 1 \end{bmatrix} \begin{bmatrix} i(n+1) \\ u(n+1) \end{bmatrix} = \begin{bmatrix} i(n) \\ u(n) \end{bmatrix} + \begin{bmatrix} \frac{T_s}{L} \\ 0 \end{bmatrix} e(n+1)$$

Thanks to the linear form of description it can be transformed to the explicit form as follows.

$$\begin{bmatrix} i(n+1) \\ u(n+1) \end{bmatrix} = \begin{bmatrix} 1 + \frac{RT_s}{L} & \frac{T_s}{L} \\ -\frac{T_s}{C} & 1 \end{bmatrix}^{-1} \begin{bmatrix} i(n) \\ u(n) \end{bmatrix} + \begin{bmatrix} 1 + \frac{RT_s}{L} & \frac{T_s}{L} \\ -\frac{T_s}{C} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{T_s}{L} \\ 0 \end{bmatrix} e(n)$$

Observe, that such inverse operation is possible only for linear systems. The nonlinear systems must operate with the implicit form of state equations.

The stability of both systems (analog and discrete) depends on the eigenvalues of state matrix A . The analog linear system is stable, when all its eigenvalues are in the left-half plane of complex plane. The stability of discrete systems requires the position of poles inside of the unit circle.

The analog system composed of passive RLC elements is stable from the definition (all eigenvalues in the left-half of the complex plane). However, stability of the discrete system depends on the type of approximation. In backward methods all eigenvalues are always inside of the unit circle, which means stability. Application of

forward Euler algorithm does not guarantee the stability. The position of eigenvalues depends on the assumed time step T_s . If it is too large the poles move outside the unit circle and the system converts to be unstable.

As an example let us take $R=1\Omega$, $L=1H$, $C=1F$ in the last example. Assuming $T_s=1,2s$ we get state matrix A in backward method as follows

$$\mathbf{A} = \begin{bmatrix} 0,275 & -0,329 \\ 0,329 & 0,604 \end{bmatrix}$$

Its eigenvalues are equal $\lambda_1=0.439+j0.287$, $\lambda_2=0.439-j0.287$. Both eigenvalues are inside of the unit circle, hence the resulting discrete system is stable. On the other side applying this step size T_s in forward algorithm we get matrix A equal

$$\mathbf{A} = \begin{bmatrix} -0,2 & -1,2 \\ 1,2 & 1 \end{bmatrix}$$

of the eigenvalues: $\lambda_1=0.4+j1.02$, $\lambda_2=0.4-j1.02$. These eigenvalues are outside the unit circle, hence, the discrete system is unstable. It can be stabilized reducing appropriately time step. For example at $T_s=0,6s$ we get the matrix A equal

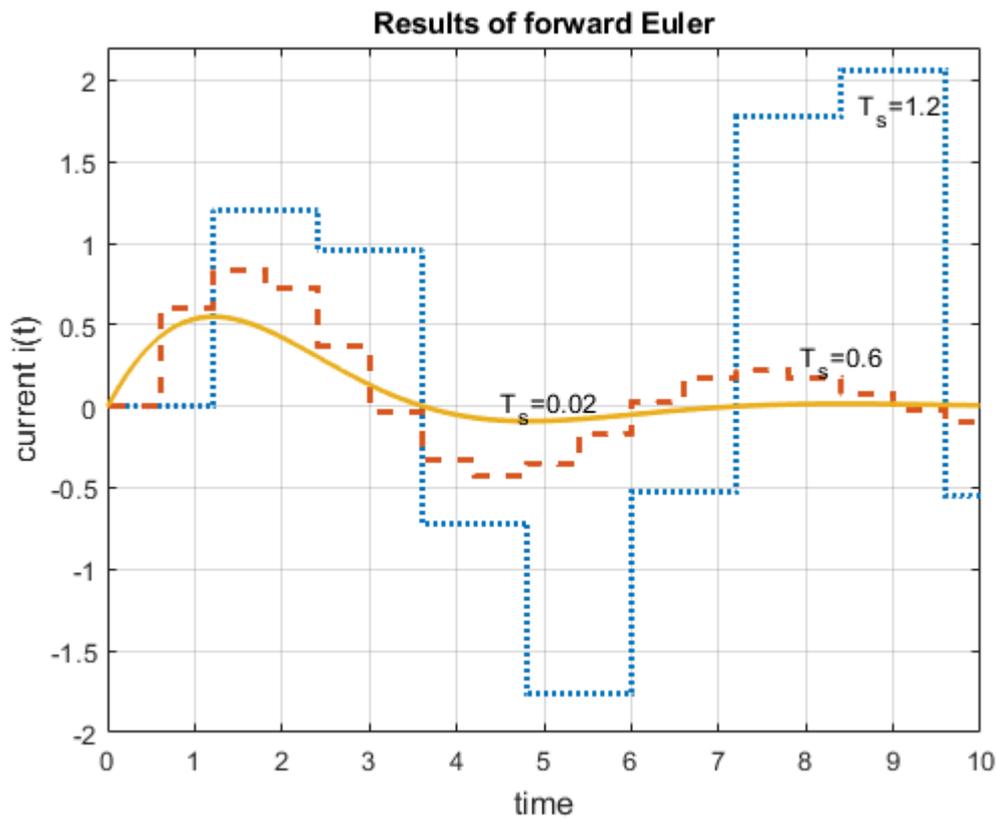
$$\mathbf{A} = \begin{bmatrix} 0,4 & -0,6 \\ 0,6 & 1 \end{bmatrix}$$

The eigenvalues of this matrix are now: $\lambda_1=-0.7+j0.52$, $\lambda_2=-0.7-j0.52$. All of them are inside of the unit circle (stable system).

The value of the discretization step T_s is important also for accuracy of approximation. Fig. 5.3 shows the results of approximation of the current $i(t)$ at different values of T_s and application of backward and forward approximations. The backward algorithm delivers the stable results irrespective of the applied sampling period. However, the accuracy of solution depends on the size of discretization step.

Different situation is in the case of forward algorithm. It might be unstable, if too long sampling period is applied. This is well illustrated in Fig. 5.3, presenting the results for $T_s=0.02s$, $0.6s$ and $1.2s$. At $T_s=1.2s$ the extrapolation (forward) algorithm has resulted in unstable solution, while the interpolation (backward) was stable, although the least accurate.

a)



b)

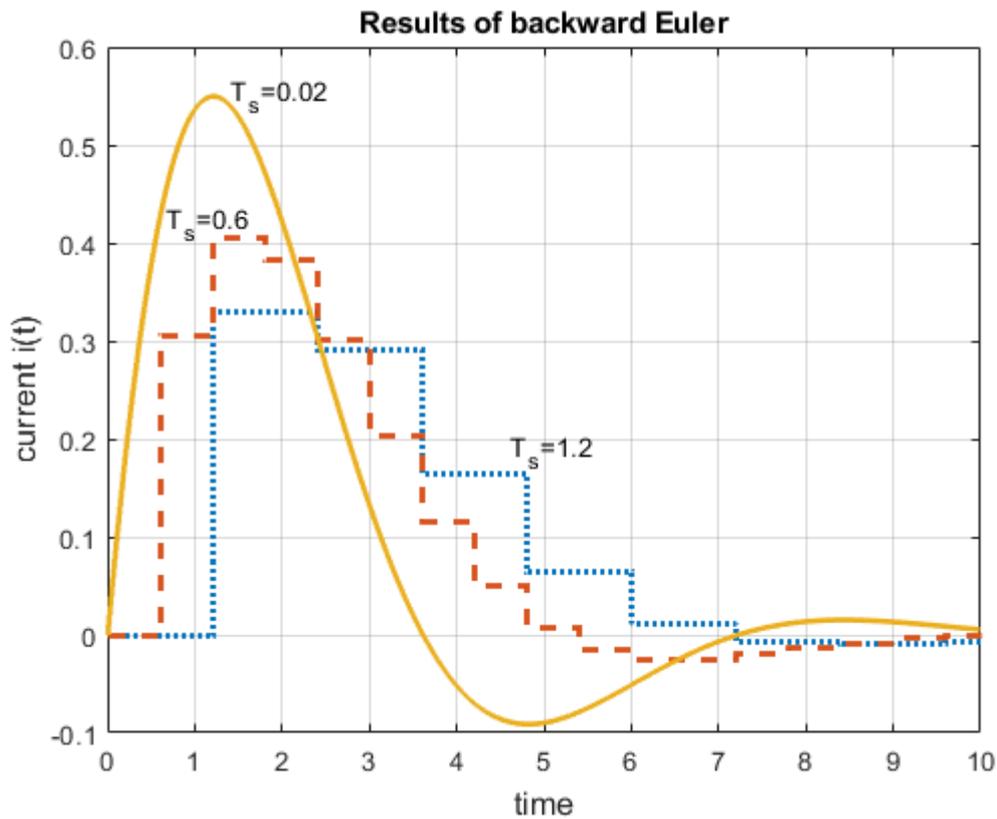


Fig. 5.3. Solution for $i(t)$ in the circuit of Fig. 5.2 at application of: a) Euler forward, b) Euler backward algorithms at different lengths of sampling period $T_s=0.021s, 0.6s$ and $1.2s$.

5.2. Z transformation in discrete system description

5.2.1. Definition of Z transformation

Z-transformation converts a **discrete-time signal**, which is a **sequence** of **real** or **complex numbers**, into a complex **frequency domain** representation. It fulfills similar role in digital systems as Laplace transformation in analog systems.

Let us consider $x(n)$ as the discrete time series defined for $n \geq 0$. Formally we define the single-sided Z-transform of this time series in the following way

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \quad (5.9)$$

The operator z^{-1} is interpreted as the unit delay operator associated with the sampling period T_s . The term z^{-k} means operation of delay by k sampling periods. The transformation (5.9) is valid only for those values of z for which the (5.9) is convergent.

As an example consider Z-transform of the signal $x(n) = a^n 1(n)$. According to the definition we have

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

Assuming $|az^{-1}| < 1$ this infinite geometric series is convergent, and its sum can be calculated from the

expression $S_n = \frac{a_0}{1 - q}$ (a_0 – the first term of series, q – the quotient). Using it we get

$$X(z) = Z(a^n) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

This equality only holds if $|z| > |a|$ (condition of convergence). In this case the region of convergence is the complex plane with a disc of radius centered at the origin and of radius a .

In the special case of $a=1$, i.e. $x(n) = 1(n)$ we get

$$Z(1(n)) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Table 5.3 presents Z-transforms for some chosen time series, often used in electrical engineering [14,23].

Table 5.3. Z-transforms of some chosen time series.

$x(n)$	$X(z)$	
$\delta(n)$	1	
$1(n)$	$\frac{z}{z-1}$	
$a^n 1(n)$	$\frac{z}{z-a}$	
$\sin(\omega T_s n)$	$\frac{z \sin(\omega T_s)}{z^2 - 2z \cos(\omega T_s) + 1}$	
$\cos(\omega T_s n)$	$\frac{z(z - \cos(\omega T_s))}{z^2 - 2z \cos(\omega T_s) + 1}$	
$e^{-\alpha T_s n} \sin(\omega T_s n)$	$\frac{ze^{-\alpha T_s} \sin(\omega T_s)}{z^2 - 2ze^{-\alpha T_s} \cos(\omega T_s) + e^{-2\alpha T_s}}$	
$e^{-\alpha T_s n} \cos(\omega T_s n)$	$\frac{z(z - e^{-\alpha T_s} \cos(\omega T_s))}{z^2 - 2ze^{-\alpha T_s} \cos(\omega T_s) + e^{-2\alpha T_s}}$	

5.2.2. Properties of the Z-transform

The properties of Z-transform are analogous to their counterparts of Laplace-transform. We will limit here only to few, the most often used in computation.

- Linearity

$$Z(ax_1(n) + bx_2(n)) = aX_1(z) + bX_2(z) \quad (5.10)$$

- Scaling in z-domain

$$Z(a^n x(n)) = X(a^{-1}z) \quad (5.11)$$

- Time delay

Assuming $x(n)=0$ for $n<0$

$$Z(x(n-k)) = z^{-k} X(z) \quad (5.12)$$

When $x(n)$ are non-zero for $n<0$

$$Z(x(n-k)) = z^{-k} X(z) + \sum_{i=1}^k x(-i)z^{-k+i} \quad (5.13)$$

- Convolution in time

The convolution in discrete system is defined as

$$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(k)x_2(n-k) \quad (5.14)$$

The Z-transform of convolution is the product of the Z-transforms of both functions

$$Z(x_1(n) * x_2(n)) = X_1(z)X_2(z) \quad (5.15)$$

- Time reversal

$$Z(x(-n))=X(z^{-1}) \quad (5.16)$$

- Decimation

$$Z(x(Kn)) = \frac{1}{K} \sum_{p=0}^{K-1} \left(z^{1/K} e^{-j2\pi p/K} \right) \quad (5.17)$$

- Value of $x(n)$ at infinity and at zero

$$\lim_{n \rightarrow \infty} x(n) = \lim_{z \rightarrow 1} (z-1)X(z) \quad (5.18a)$$

$$x(0) = \lim_{n \rightarrow \infty} X(z) \quad (5.18b)$$

5.2.3. Inverse Z-transform

The Z-transform is an invertible operation. On the basis of $X(z)$ we can reconstruct the original time series. The formal definition of inverse transformation is given in the form

$$x(n) = Z^{-1}[X(z)] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} dz \quad (5.19)$$

where C is a counterclockwise closed path encircling the origin and entirely in the [region of convergence](#). However, direct application of this definition is very difficult, since in most cases of electrical engineering the function $X(z)$ is complex (often rational). Therefore some other methods are applied. We limit here to the method based on residue theorem.

Let us assume that $X(z)$ is a rational function of the form

$$X(z) = \frac{L(z)}{M(z)} = \frac{b_M z^M + b_{M-1} z^{M-1} + \dots + b_1 z + b_0}{b_N z^N + b_{N-1} z^{N-1} + \dots + b_1 z + b_0} \quad (5.20)$$

This function possesses the poles which might be finite or infinite, complex or real, single or multiple. According to the residue theorem the original time series $x(n)$ may be calculated from the expression

$$x(n) = \sum_k \text{res}(X(z)z^{n-1})|_{z=z_k} \quad (5.21)$$

The residues are calculated over all poles z_k ($k=1, 2, \dots, N$) of the function. They are defined by the following expressions [16]

- For single pole z_k

$$\text{res}(X(z)z^{n-1})|_{z=z_k} = \lim_{z \rightarrow z_k} [X(z)(z-z_k)z^{n-1}] \quad (5.22)$$

- For multiple pole z_k of the multiplicity l

$$\operatorname{res}(X(z)z^{n-1})|_{z=z_k} = \frac{1}{(l-1)!} \lim_{z \rightarrow z_k} \frac{d^{l-1}}{dz^{l-1}} [X(z)(z-z_k)^l z^{n-1}] \quad (5.23)$$

Example 5.3

The procedure of inverse Z-transformation will be illustrated on the example of $X(z)$ given as follows

$$X(z) = \frac{z}{z^2 - 2,5z + 1} = \frac{z}{(z - 0,5)(z - 2)}$$

This function has two poles: $z_1=0,5$ and $z_2=2$. Hence, the original function is the sum of two residues

$$x(n) = \operatorname{res}(X(z)z^{n-1})|_{z=0,5} + \operatorname{res}(X(z)z^{n-1})|_{z=2} = \frac{z^n}{(z-2)}|_{z=0,5} + \frac{z^n}{(z-0,5)}|_{z=2} = -\frac{2}{3} \left(\frac{1}{2}\right)^n + \frac{2}{3} 2^n$$

The original function description contains two terms as a function of discrete time. Both depend on the values of poles. Observe that the first term decreases with time to zero and the second increases to infinity with t tending to infinity.

Example 5.4

The second example illustrates the case with double pole. Function $X(z)$ is given in the form

$$X(z) = \frac{z}{(z - 0,5)^2}$$

Taking into account the double pole at $z_1=0,5$ we get the inverse Z-transform in the form

$$x(n) = \operatorname{res}(X(z)z^{n-1})|_{z=0,5} = \lim_{z \rightarrow z_k} \frac{d}{dz} [X(z)(z - 0,5)^2 z^{n-1}] = \frac{d}{dz} z^n = n z^{n-1} = 2n \left(\frac{1}{2}\right)^n$$

In the case of complex poles the residue method requires tedious complex computations. In such case it is easier to decompose the given function into first and second order partial fractions, transform them to the forms, which appear in the table of Z-transforms and read the original time functions directly from this table.

Example 5.5

This will be illustrated by the following example

$$X(z) = \frac{5z}{z^2 - z + 1}$$

This expression should be transformed to the form appearing in the fourth row of table 5.3. After making proper rearrangements we get

$$X(z) = \frac{5z}{z^2 - 2z \cdot 0,5 + 1}$$

It is evident now, that $\cos(\omega T_s) = 0,5 \rightarrow \omega T_s = \pi/3$. The numerator should contain the term

$$\sin(\omega T_s) = \sqrt{3/4}$$

. Hence the final transformed expression for $X(z)$ takes the form

$$X(z) = \frac{5\sqrt{4/3} \cdot z\sqrt{3/4}}{z^2 - 2z \cdot 0,5 + 1}$$

Comparing it with the prototype form from the fourth row of table 5.3 we get the solution for $x(n)$ in the form

$$x(n) = 5\sqrt{4/3} \sin(\pi n/3)$$

Example 5.6

The next example of $X(z)$ is in the form

$$X(z) = \frac{6z}{z^2 - 3z + 4}$$

The prototype of this function in table 5.3 is the form appearing in the sixth row. We have to transform the given function to the form corresponding to this row. Therefore,

$$X(z) = \frac{6z}{z^2 - 2z \cdot 2 \cdot 3/4 + 2^2}$$

The identification of the elements of the denominator shows that $e^{-2\alpha T_s} = 2^2 \rightarrow e^{-\alpha T_s} = 2$, $\cos(\omega T_s) = 3/4 \rightarrow \sin(\omega T_s) = \sqrt{7}/4 \rightarrow \omega T_s = 0,7227rad$. In such situation we may present the given function in the form

$$X(z) = \frac{z \cdot 2 \cdot \sqrt{7}/4 \cdot (3 \cdot 4/\sqrt{7})}{z^2 - 2z \cdot 2 \cdot 3/4 + 2^2}$$

from which we identify the original time series $x(n)$ as

$$x(n) = \frac{12}{\sqrt{7}} 2^n \sin(0,7227n)$$

5.2.4. Transfer function of discrete system

Transfer function of discrete system is defined as the ratio of Z-transform of the response signal $Y(z)$ to the Z-transform of the input excitation $X(z)$

$$H(z) = \frac{Y(z)}{X(z)} \quad (5.24)$$

This function $H(z)$ represents also the Z-transform of the impulse response $h(n)$ of the discrete system, i.e.,

$$H(z) = Z[h(n)] \quad (5.25)$$

It follows from this

$$h(n) = Z^{-1}[H(z)] \quad (5.26)$$

In general, transfer function $H(z)$ takes the rational form of the numerator $L(z)$ and denominator $M(z)$. Standard form of presentation of this function used in digital signal processing operates with the negative powers of z . Two different forms are recognized in such case [14,15].

- System of Finite duration of Impulse Response (FIR)

$$H(z) = L(z) = b_0 + b_1 z^{-1} + \dots + b_M z^{-M} \quad (5.27)$$

- System of Infinite duration of Impulse Response (IIR)

$$H(z) = \frac{L(z)}{M(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}} \quad (5.28)$$

Each of these forms might be associated with the corresponding structure of digital circuit. In the case of FIR the transfer function defined as $H(z)=Y(z)/X(z)$ can be presented as follows

$$Y(z) = b_0X(z) + b_1z^{-1}X(z) + \dots + b_Mz^{-M}X(z) \quad (5.29)$$

Taking into account the time interpretation of the operator z^{-1} as the unit delay, we may write

$$y(n) = b_0x(n) + b_1x(n - 1) + \dots + b_Mx(n - M) \quad (5.30)$$

The structure of digital circuit corresponding to it is presented in Fig. 5.4. The building elements of this circuit are: the unit delay blocks represented by z^{-1} , the blocks of gains described by polynomial coefficients b_i ($i = 0, 1, \dots, M$) and the summing block, adding the signals.

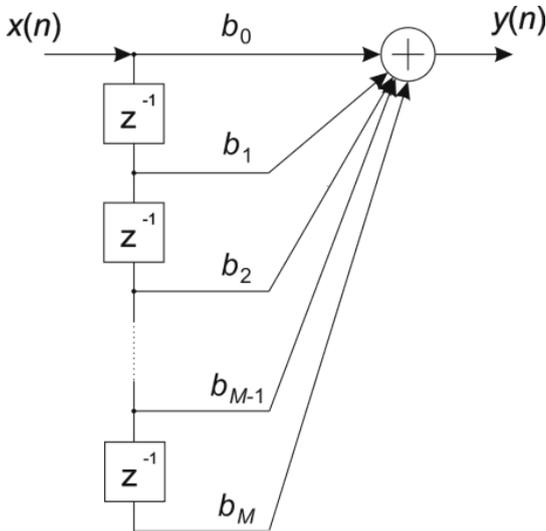


Fig. 5.4. FIR discrete system structure.

In the case of IIR system we do similar operations. However, this time we have two polynomials $L(z)$ and $M(z)$. Therefore, on the basis of (5.28) we may write

$$Y(z) + a_1z^{-1}Y(z) + \dots + a_Nz^{-N}Y(z) = b_0X(z) + b_1z^{-1}X(z) + \dots + b_Mz^{-M}X(z) \quad (5.31)$$

(5.31)

Transforming this expression to the time form we get

$$y(n) = b_0x(n) + b_1x(n - 1) + \dots + b_Mx(n - M) - (a_1y(n - 1) + \dots + a_Ny(n - N)) \quad (5.32)$$

(5.32)

This equation may be represented by the circuit structures presented in Fig. 5.5. The figure on the left presents the non-canonical structure and on the right – the canonical one (minimum number of delay blocks).

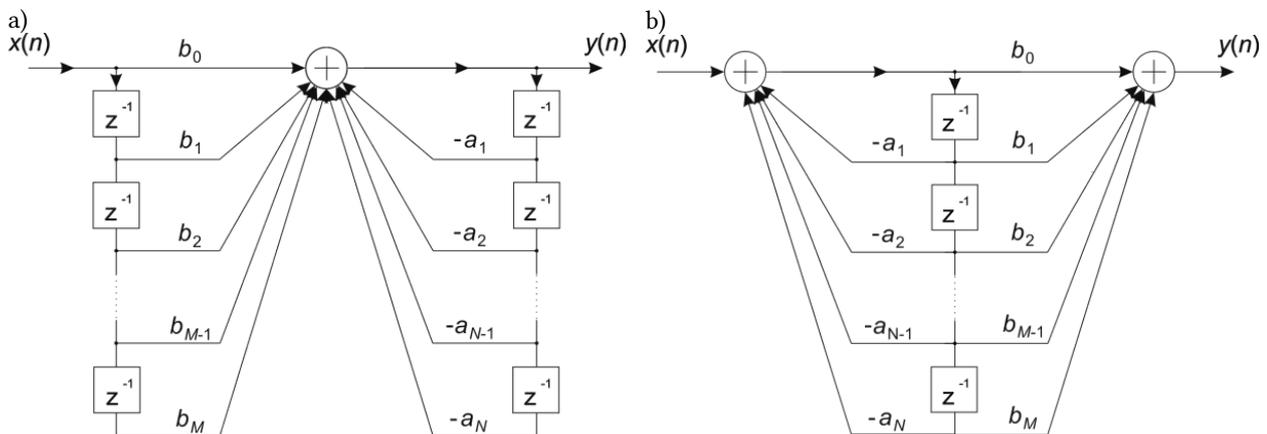


Fig. 5.5. Structure of IIR system: a) non-canonical with respect to the number of unit delays, b) canonical.

Canonical structure operates with the minimum number of unit delay blocks. Its number is equal to $\max(N,M)$. The significant difference between FIR and IIR structures is the presence of feedback in IIR, following from the existence of the denominator $M(z)$. The FIR system does not possess the denominator. Hence, there are no poles, which might appear outside the unit circle. Therefore, the FIR circuit is always stable.

The IIR system possesses the poles (roots of the denominator), therefore, its stability depends on their positions. Let us consider Bounded Input – Bounded Output (BIBO) stability of the system. In this respect the system is stable when at limited excitation $x(n)$ the response of the system is always limited for any point of time. From the definition of transfer function it follows that $Y(z)=H(z)X(z)$. It means the time response can be calculated from the formula

$$y(n) = h(n) * x(n) \tag{5.33}$$

in which $h(n) = Z^{-1}(H(z))$ is the impulse response (response of circuit to the excitation $\delta(n)$).

The signal $x(n)$ is limited by the definition. Therefore, the limitation of response $y(n)$ depends only on $h(n)$. The time form of $h(n)$ depends on the positions of poles z_k of the transfer function $H(z)$. Observe, that inversed form of $H(z)$ depends on z_k^n . Assuming $z_k=a_k$, from the transfer function

$$H(z) = \sum_k \frac{A_k z}{z - a_k} = \sum_k \frac{A_k}{1 - a_k z^{-1}} \tag{5.34}$$

for single poles we have

$$h(n) = \sum_k A_k a_k^n \tag{5.35}$$

In the case of multiple poles of multiplicity l there appears also the term $\frac{A_k z}{(z - a_k)^l}$, which corresponds in time domain to the additional component

$$h_k(n) = A_k \frac{n(n - 1) \cdots (n - l + 2)}{(l - 1)!} a_k^{n-l+1} \tag{5.36}$$

As it is seen, in discrete time systems the limitation of succeeding terms of the impulse response (condition of stability) depends on position of poles and their multiplicity. System is asymptotically stable (the impulse response tends to zero) when all poles are located inside of the unit circle $|z| < 1$. If any pole is outside this circle the discrete system is automatically unstable (the response tends to infinity with time). In the case of poles placed directly on the unit circle the stability depends on their multiplicity. If they are single, the system is stable but not asymptotically (the impulse response is always limited, but not zero). If the poles on the unit circle are multiple the discrete system is unstable.

Transfer function of discrete system allows defining also the frequency characteristics. Frequency characteristics correspond to the frequency along the unit circle, i.e. for

$$z = e^{j\omega T_s} = \cos(\omega T_s) + j \sin(\omega T_s) \tag{5.37}$$

. Frequency characteristics are then defined as

$$H(\omega) = H(z = e^{j\omega T_s}) \tag{5.37}$$

Absolute value of this function represents the magnitude characteristics and angle – the phase characteristic. Both are functions of frequency.

5.3. Transformation of analog transfer function to discrete form

There are many methods of transformation of the transfer function $H(s)$ of analog system into transfer function

$H(z)$ of the discrete system. The most often used in practice are two transformations. One is called the bilinear and the second invariance of the impulse response [14,16]. Both preserve the order of original analog system.

5.3.1. Bilinear transformation

Bilinear transformation is based on the trapezoidal method of integration of the differential equation $dx/dt=f(t)$. According to the trapezoidal method, solution of $x(t)$ in the discrete time point $t=nT_s$ can be written in the form

$$x_n = x_{n-1} + \frac{T_s}{2} \left[\frac{dx_n}{dt} + \frac{dx_{n-1}}{dt} \right] \quad (5.38)$$

In continuous time description the derivative is represented by the operator s , and in discrete time the unit delay operation is associated with the operator z^{-1} . Therefore, the eq. (5.38) may be written using these operators.

$$x_n = z^{-1}x_n + \frac{T_s}{2} [sx_n + sz^{-1}x_n] \quad (5.39)$$

From this we get

$$z = 1 + \frac{T_s}{2} [sz + s] \quad (5.40)$$

Taking into account $f_s=1/T_s$ we can present the above equations in a form allowing direct transformation $H(s)$ into $H(z)$

$$s = 2f_s \frac{1 - z^{-1}}{1 + z^{-1}} \quad (5.41)$$

It is possible to return from $H(z)$ to $H(s)$ by writing z as a function of operator s

$$z = \frac{1 + \frac{T_s}{s}s}{1 - \frac{T_s}{s}s} \quad (5.42)$$

The choice of the values of f_s should satisfy the Nyquist theorem (the sampling rate f_s should be higher than double value of the maximum frequency f_{max} existing in the signal). Determination of f_{max} requires the additional frequency analysis of the original analog system.

Bilinear transformation provides stable mapping the continuous form of system description into discrete form. It means, that the poles of analog system in the left half-plane are moving to the inside of the unit circle. However, the frequency transformation is of nonlinear character. The relation between the angular frequency ω_a of the analog system and frequency ω of the discrete system is nonlinear. To show it let us assume the discrete system frequency characteristic defined for $z = e^{j\omega T_s}$. Putting it into (5.41) we get

$$s = 2f_s \frac{1 - e^{-j\omega T_s}}{1 + e^{-j\omega T_s}} = j2f_s \cdot tg\left(\frac{\omega T_s}{2}\right) = j\omega_a \quad (5.43)$$

where ω_a represents the frequency on continuous time (analog) system. It means, that the unit circle described by $e^{j\omega T_s}$ in discrete space is directly converted to imaginary axis of the complex variable $s = j\omega_a$. This mapping is described by

$$\omega_a = 2f_s \cdot tg\left(\frac{\omega T_s}{2}\right) \quad (5.44)$$

The bilinear transformation changes the frequency in a tangensoidal way. Only for very small values of frequency the relation is close to the linear one, since in such case $tg(\alpha) \approx \alpha$ (in radial measure).

5.3.2. Method of invariance of impulse response

In this method we assume that impulse response in sampling points of both (analog and discrete) systems should be equal. It means, that in sampling points we have

$$h(n) = h(t)|_{t=nT_s} \quad (5.45)$$

The original transfer function $H(s)$ can be presented in the form of partial fraction expansion using the poles p_i of the system (only single poles are assumed here)

$$H(s) = \sum_{i=1}^N \frac{r_i}{s - p_i} \quad (5.46)$$

The impulse response is then equal

$$h(t) = \sum_{i=1}^N r_i e^{p_i t} \quad (5.47)$$

The equality of (5.45) requires

$$h(n) = \sum_{i=1}^N r_i e^{p_i n T_s} \quad (5.48)$$

Transfer function $H(z)$ of discrete system is the Z-transform of $h(n)$. Hence, from the definition of Z-transformation we can write

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} \left(\sum_{i=1}^N r_i e^{p_i n T_s} \right) z^{-n} = \sum_{i=1}^N r_i \sum_{n=0}^{\infty} (e^{p_i T_s} z^{-1})^n = \sum_{i=1}^N \frac{r_i}{1 - z^{-1} e^{p_i T_s}} \quad (5.49)$$

Observe, that the term $\frac{r_i}{s - p_i}$ of $H(s)$ is changed to $\frac{r_i}{1 - z^{-1} e^{p_i T_s}}$ in discrete form of description. The pole p_i of analog system has been converted to the pole $z_i = e^{p_i T_s}$ of discrete one. Assuming stable analog system (p_i in the left half-plane) we always get $|z_i| < 1$, and hence the condition of stability of discrete system is fulfilled.

Example 5.7

As an example consider the transfer function of analog system given as follows

$$H(s) = \frac{2s}{s^2 + 3s + 2} = \frac{2s}{(s + 1)(s + 2)}$$

It should be converted to partial fraction form given below

$$H(s) = \frac{-2}{s + 1} + \frac{4}{s + 2}$$

Application of eq. (5.49) allows getting the form of $H(z)$

$$H(z) = \frac{-2}{1 - z^{-1} e^{-T_s}} + \frac{4}{1 - z^{-1} e^{-2T_s}} = \frac{2 + z^{-1} (e^{-2T_s} - 4e^{-T_s})}{1 - z^{-1} (e^{-T_s} + e^{-2T_s}) + z^{-2} e^{-3T_s}}$$

To obtain the purely numerical form of $H(z)$ we have to find proper value of T_s . Let us assume the condition for f_{max} in the form

$$|H(j\omega_{max})| < 0.01$$

Applying $s = j\omega_{max}$ to the transfer function $H(s)$ we get

$$\left| \frac{2j\omega_{\max}}{(2 - \omega_{\max}^2) + j3\omega_{\max}} \right| < 0.01$$

Solving it with respect to frequency we get $\omega_{\max}=200$ and $f_{\max}=31.8$. Let us assume $f_s=100$ and $T_s=0.01$. Substituting it into $H(z)$ we get final numerical form of discrete transfer function.

$$H(z) = \frac{2 - 2.9801z^{-1}}{1 - 1.9702z^{-1} + 0.9704z^{-2}}$$

It is easy to check that both poles of $H(z)$ are inside of the unit circle.

Program Matlab has many functions devoted to design of discrete systems, called further discrete filters. The basic functions applying analog prototypes in design of discrete (digital) filters include: `butter.m` (Butterworth filter), `cheby1.m` and `cheby2.m` (Chebyshev filters) and `ellip.m` (elliptic filters). Moreover, there is the graphic interface for designing many types of discrete filters, which is called `FilterDesigner`.

5.4 Project devoted to filter design

In one of the projects student should design the particular discrete filter of the specifications given by the teacher and applying manually all steps of design procedure based on analog prototype. Moreover, as a part of the project student should apply also the interface `FilterDesigner` [12] to design the proper band-stop filter to denoise the Haendel music corrupted by narrow band random noise (for example the file 'handel_szum1.WAV' accessible in ISOD).

An example of the design procedure the following task is formulated.

Part I: Design of discrete elliptic BP filter using analog prototype.

Filter specifications:

$$R_p=1 \text{ dB}$$

$$R_s=40 \text{ dB}$$

$$\Omega_p=[2000 \text{ Hz } 4000 \text{ Hz}]$$

$$\Omega_s=[1800 \text{ Hz } 4200 \text{ Hz}]$$

%Matlab program:

%Analog filter

`rp=1;`

`rs=40;`

`fp=[2000, 4000]; fstop=[1800, 4200];`

`fs=20000; fNy=fs/2;`

% Order of filter

`[n,wn]=ellipord(fp/fNy, fstop/fNy,rp, rs);`

`[ba,aa]=ellip(n,rp,rs,wn,'s')`

%Frequency characteristics

`wa=0:0.01:1;`

`ha=freqs(ba,aa,wa);`

`figure(1)`

`subplot(2,1,1),plot(wa*fNy,abs(ha)); grid % Linear plot`

```

axis([0 10000 0 1]); ylabel('Magnitude - linear')
title('Analog filter')
subplot(2,1,2),plot(wa*fNy,20*log10(abs(ha)));
grid % Linear plot
axis([0 10000 -50 0]);
xlabel('Frequency Hz'), ylabel('Magnitude - logarithmic')

% Digital filter using bilinear transformation
[bd,ad]=bilinear(ba,aa,1)

%Frequency characteristics
wd=0:0.001:1;
[hd,wd]=freqz(bd,ad,wd);
figure(2)
subplot(2,1,1), plot(wd*fNy,abs(hd)), grid % Linear plot
axis([0 10000 0 1]); ylabel('Magnitude - linear')
title('Discrete filter')
subplot(2,1,2), plot(wd*fNy,20*log10(abs(hd))),
grid % Logarithmic plot
axis([0 10000 -50 0]);
xlabel('Frequency Hz'), ylabel('Magnitude - logarithmic')

```

Part II: Using *filterDesigner* of Matlab design band-stop filter (elliptic) to denoise the music signals. Apply the following procedure:

1) Read the noisy music signals:

```

[y,fs]=audioread('handel_szum1.WAV');
soundsc(y);

```

2) Determine frequency characteristic of the signal using the following command lines:

```

[pxx, f]=pwelch(y,2048,512,[],fs,'onesided')
plot(f,pxx), grid

```

3) On the basis of frequency characteristic determine the required band-stop range of band-stop filter to remove the noise

4) Apply *filterDesigner* to design proper elliptic band-stop filter of the estimated band-stop range and $R_p=1\text{dB}$, $R_s=60\text{dB}$

5) Convert filter parameters to single section (menu: Edit->Convert to Single Section) and export to workspace of Matlab (menu File->Export (Coefficients)).

6) Filter the noisy signal and listen to the denoised signals using the command

```
z=filter(Num,Den,y)
soundsc(z)
```

5.4. Switched capacitor circuits

5.4.1. Basic notions

Switched capacitors circuits (shortly SC) represent specific form of implementation of discrete systems in an integrated form [22]. A switched capacitor works by moving charges into and out of capacitors when switches are opened and closed. The circuits operate with only capacitors cooperating with the switches controlled in two opposite phases of the sampling period. One half of the switching cycle is treated as an even (positive) and the second as an odd (negative) phase. Both phases are not overlapping. The graphical interpretation of the signals controlling the switches within the switching cycle is depicted in Fig. 5.6.

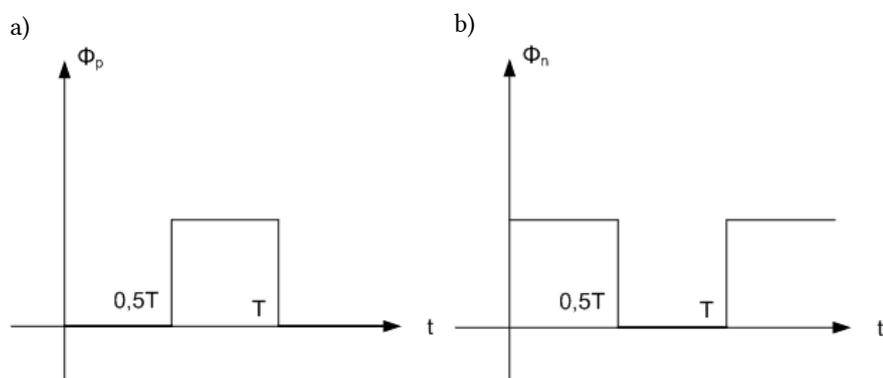


Fig. 5.6. The signals controlling the switches: a) Φ_p – even phase, b) Φ_n – odd phase.

The even phase will be denoted by us with the upper index p , and odd with the index n . The graphical symbols of switches closed in even (index p) and odd (index n) phases are presented in Fig. 5.7.

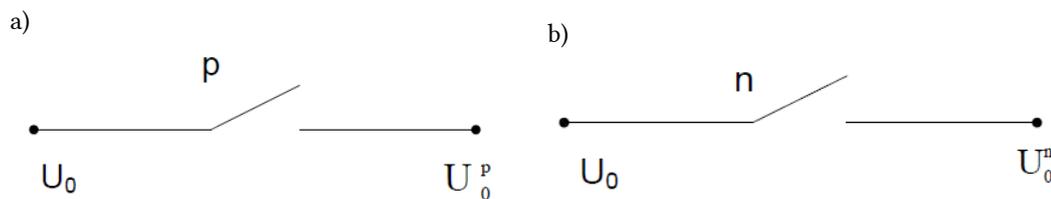
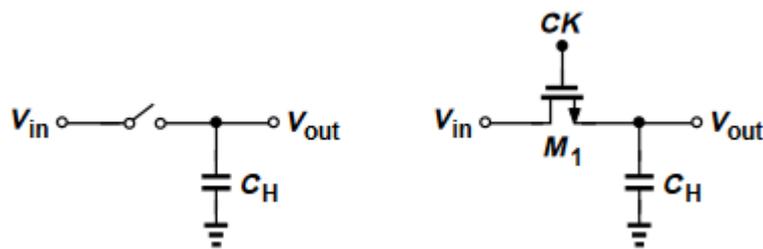


Fig. 5.7. Symbols of switches connecting the input voltage U_0 to the right terminal in a) even phase, b) odd phase.

The switches are easy to implement in the integrated technology in the form of field effect transistors. An example of it is shown in Fig. 5.8.



a) b)

Fig. 5.8 The MOS implementation of switch in SC circuit: a) SC circuit structure, b) implementation of switch by a MOS device.

Proper arrangement of capacitors and configuration of switches allows implementing the effect of resistors, with the resistance dependent on the value of capacitance and frequency of controlling signal. This makes them more suitable for use within **integrated circuits**, where accurately specified resistors and capacitors are not economical to construct.

Observe, that the signal is switched from one phase to the second one every half of the switching cycle and may be represented mathematically using the delay operator $z^{-1/2}$. The output signal may be received in either even or odd phase. The same may be done with the input (source) signal.

The structures supplying the circuits in the even (p) and odd (n) phase are presented schematically in Fig. 5.9.

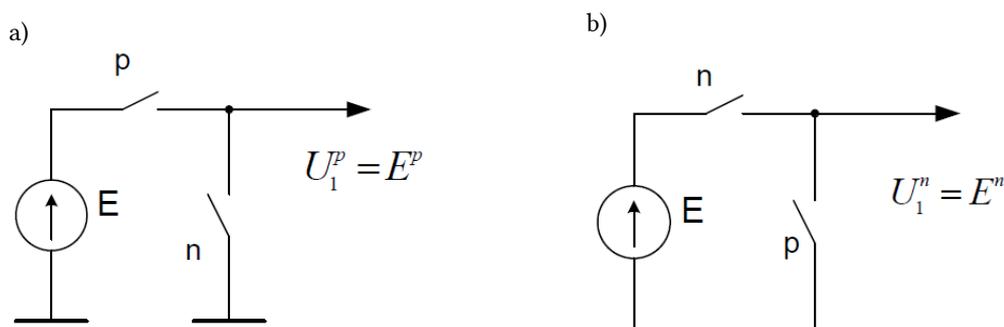


Fig. 5.9. The structures of supplying the S.C. circuit in a) even and b) odd phase.

Let us denote the output voltage signal by U_0 and input source by U_1 . Taking into account even and odd phases of operation of SC circuits we can define 4 types of transfer function.

$$H_{pp}(z) = \frac{U_0^p}{U_1^p} \quad (5.50)$$

$$H_{pn}(z) = \frac{U_0^p}{U_1^n} \quad (5.51)$$

$$H_{np}(z) = \frac{U_0^n}{U_1^p} \quad (5.52)$$

$$H_{nn}(z) = \frac{U_0^n}{U_1^n} \quad (5.53)$$

They may be also presented in a compact matrix form

$$\begin{bmatrix} U_0^p \\ U_0^n \end{bmatrix} = \begin{bmatrix} H_{pp}(z) & H_{pn}(z) \\ H_{np}(z) & H_{nn}(z) \end{bmatrix} \begin{bmatrix} U_1^p \\ U_1^n \end{bmatrix} \quad (5.54)$$

These individual transfer functions differ only by the time delay associated with the switching moments.

5.4.2. Circuit models of SC circuits

Switched-capacitor circuits are analyzed by writing down charge conservation equations and solving them with a computer algebra tool. For hand analysis and for getting more insight into the circuits, it is also possible to apply the signal flow graphs, with a method that is very similar for switched-capacitor and continuous-time circuits.

In each arrangement of switches and capacitor the flow of charge within the capacitor is analyzed and its change determined. The change of charge related to the time is interpreted as the current. In this way the resistive SC models of the circuit can be built [16,22].

Example 5.8

The simplest switched-capacitor circuit is the switched-capacitor resistor, made of one capacitor C and two switches denoted symbolically by p (even phase) and n (odd phase) which connect the capacitor with a given frequency alternately to the input and output of the external circuit (Fig. 5.10). Each switching cycle transfers a charge q from the input to the output at the switching frequency f . The charge q on a capacitor C with a voltage V between the plates is given by:

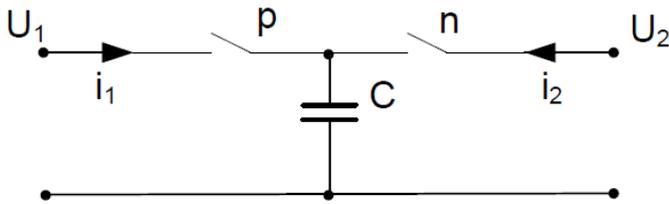


Fig. 5.10. The SC circuit with two opposite-phase switches.

The changes of charge of the capacitor in even and odd phases are described by the equations

$$\begin{aligned} \Delta q_1^p &= CU_1^p - z^{-1/2} CU_2^n \\ \Delta q_2^n &= CU_2^n - z^{-1/2} CU_1^p \end{aligned} \quad (5.55)$$

Because this charge is transferred at a rate f , the rate of transfer of charge per unit time represents current $i=fq$ (the continuous transfer of charge from one node to another is equivalent to a current). The changes of the charge are equivalent to the currents flowing in the even and odd phases within the switching cycle T with $f=1/T$. Dividing the changes of charge by the period T we get the equivalent currents in both phases [16,22]

$$\begin{aligned} i_1^p &= CfU_1^p - z^{-1/2} CfU_2^n \\ i_2^n &= CfU_2^n - z^{-1/2} CfU_1^p \end{aligned} \quad (5.56)$$

The first term in both equations represents the resistor of the conductance $G = Cf$. The second term represents some memory from the previous phase and is a form of voltage controlled current source, called storistor. The controlling coefficient is called storistance $G_s = Cfz^{-1/2}$. The equation (5.56) can be associated with the circuit model presented in Fig. 5.11. The currents of the terminals are defined only for even phase (input terminal) and odd phase (output terminal).

Example 5.9

The second example is the SC circuit with four switches connecting the capacitor C to four different terminals of the external circuit denoted by the voltages: U_1, U_2, U_3 and U_4 as is shown in Fig. 5.12.

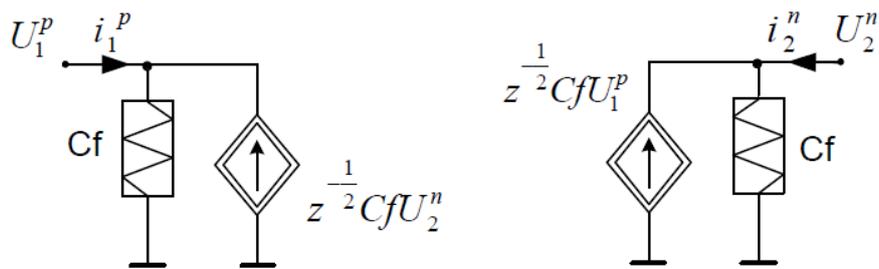


Fig. 5.11. Storistor-resistor model of the SC circuit of Fig. 5.10.

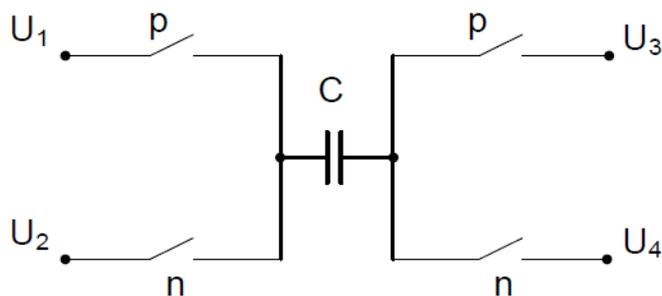


Fig. 5.12. Switched capacitor circuit with 4 switches.

The changes of capacitor charge in both phases of switching cycle and the currents following from it are given by

$$\begin{aligned} \Delta q_1^p &= CU_{13}^p - z^{-1/2} CU_{24}^n \rightarrow i_1^p = CfU_{13}^p - z^{-1/2} CfU_{24}^n \\ \Delta q_2^n &= CU_{24}^n - z^{-1/2} CU_{13}^p \rightarrow i_2^n = CfU_{24}^n - z^{-1/2} CfU_{13}^p \end{aligned} \quad (5.52)$$

The equivalent resistor-storistor model of this circuit is presented in Fig. 5.13.

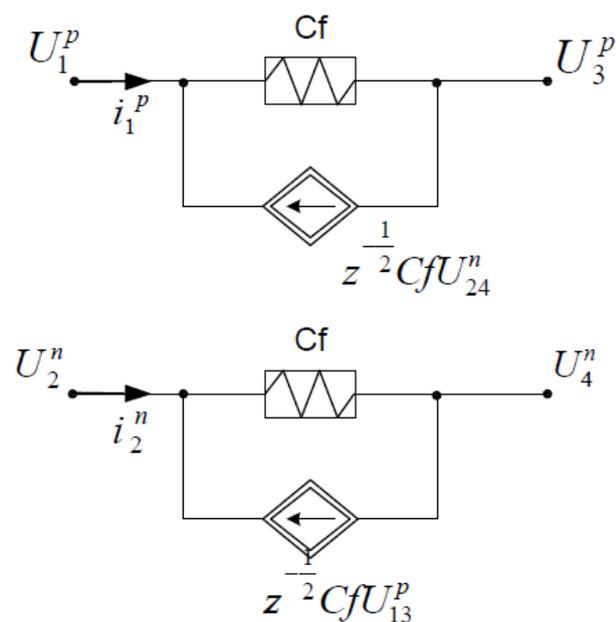


Fig. 5.13. Storistor-resistor model of the SC circuit of Fig. 5.12.

Example 5.10

Fig. 5.14 show the structure of capacitor with two switches, both controlled in even phase. In the odd phase the capacitor is isolated.

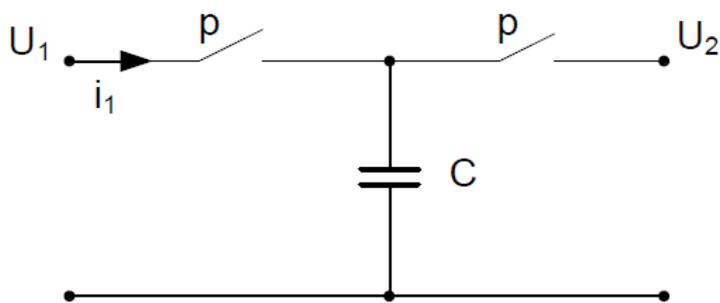


Fig. 5.14. The capacitor with two switches turned on only in even phase.

The change of charge and currents flowing in even phase through capacitor are as follows

$$\Delta q_1^p = CU_1^p - z^{-1}CU_1^p \rightarrow i_1^p = (Cf - z^{-1}Cf)U_1^p \quad (5.53)$$

The equivalent model of the SC circuit is presented in Fig. 5.15. It takes the form of parallel connection of resistor of the conductance $G=Cf$ and the storistor controlled by its own voltage (described by the admittance $Y(z)=-z^{-1}Cf$). In an odd phase the flow of signals between input and output terminals of SC circuit stops.

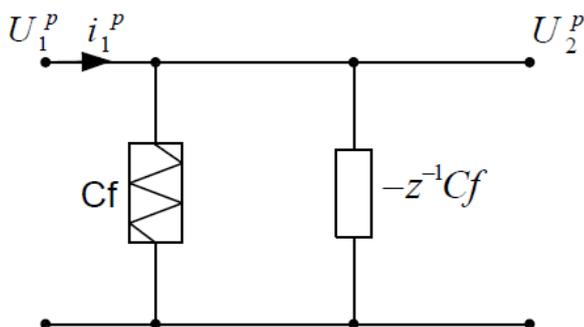


Fig. 5.15. Model of SC circuit of Fig. 5.14.

Example 5.11

Fig. 5.16 presents the grounded capacitor which is short circuit by the switch in odd phase.

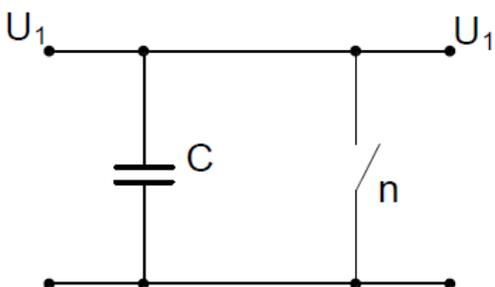


Fig. 5.16. Capacitor with one switch closed in odd phase.

From the change of charge of the capacitor we get

$$\begin{aligned}\Delta q_1^p &= CU_1^p - 0 \rightarrow i_1^p = CfU_1^p \\ \Delta q_1^n &= 0 - z^{-1/2}CU_1^p \rightarrow i_1^n = -z^{-1/2}CfU_1^p\end{aligned}\quad (5.54)$$

The resistor-storistor model of this circuit is presented in Fig. 5.17. The circuit represents the conductance $G=Cf$ in even phase and short circuit in odd phase.

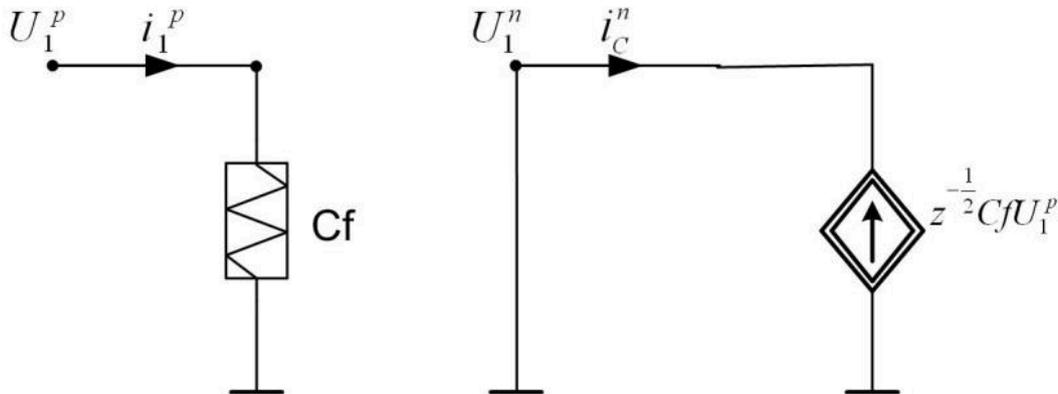


Fig. 5.17. The SC model of the circuit of Fig. 5.16.

Example 5.12

Fig. 5.18 shows the structure of SC circuit composed of floating capacitor and two switches, operating in even and odd phases.

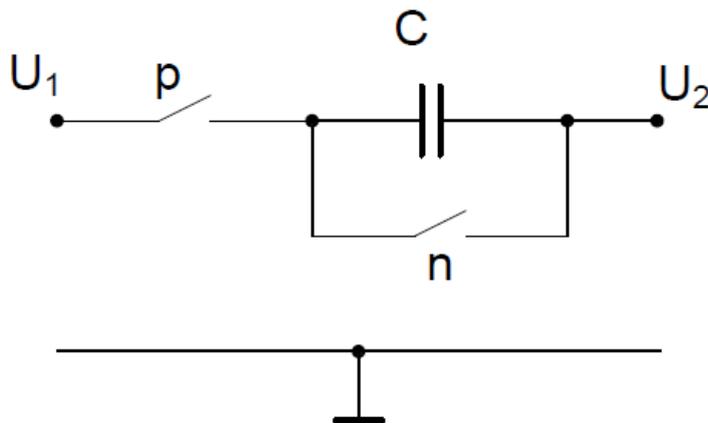


Fig. 5.18. Floating capacitor with two switches operating in different phases.

From the equations describing the change of charge of capacitor we get

$$\begin{aligned}\Delta q_1^p &= CU_{12}^p - 0 \rightarrow i_1^p = CfU_{12}^p \\ i_1^n &= 0, \quad i_2^n = 0\end{aligned}\quad (5.55)$$

The model of this circuit is presented in Fig. 5.19. The floating resistor of the conductance $G=Cf$ is implemented in even phase. In odd phase we observe the break in transmission of signals.

Example 5.13

Fig. 5.20 presents typical symmetrical structure of SC circuit, containing 4 switches and implementing the resistance in even phase.

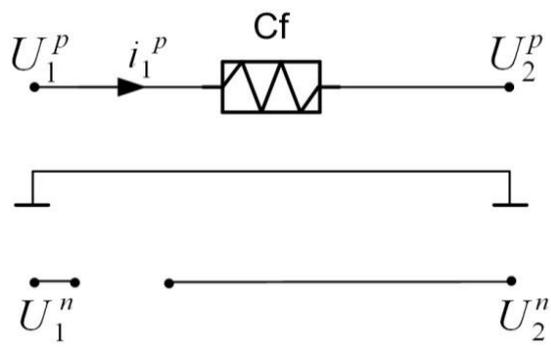


Fig. 5.19. Model of the SC circuit of Fig. 5.18.

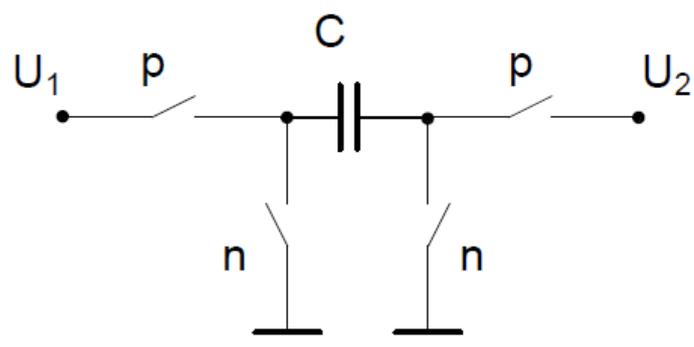


Fig. 5.20. The symmetrical SC circuit structure with four switches implementing the resistance.

From the equations describing the change of charge of capacitor we get

$$\begin{aligned} \Delta q_1^p &= CU_{12}^p - 0 \rightarrow i_1^p = CfU_{12}^p \\ \Delta q_1^n &= 0 \rightarrow i_1^n = 0 \end{aligned} \tag{5.56}$$

It is evident, that the circuit of Fig. 5.21 in even phase implements the pure floating resistor of conductance $G=Cf$. In odd phase the circuit represents the break.

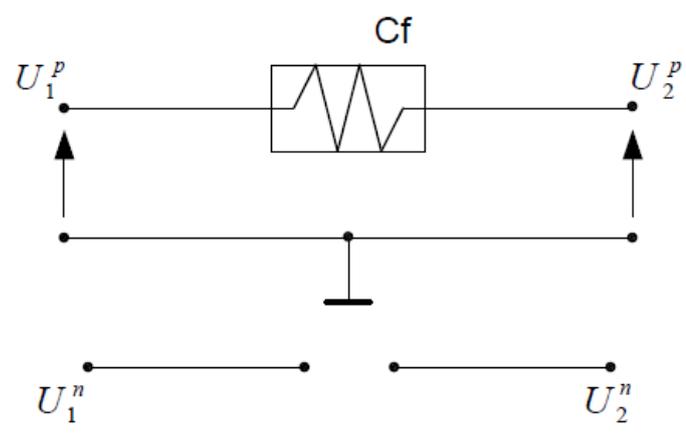


Fig. 5.21. Resistive model of SC circuit of Fig. 5.20.

Two next examples represent the capacitors, deprived of any switches. Elements of this type of connection are also present in the SC circuits.

Example 5.14

Fig. 5.22 shows the grounded capacitor.

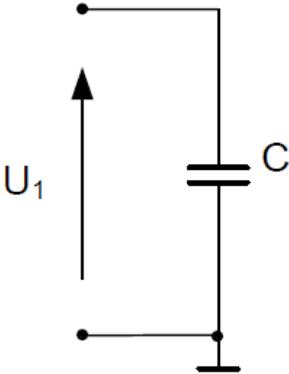


Fig. 5.22. The grounded capacitor.

The changes of charge of the capacitors in both phases are described by

$$\begin{aligned} \Delta q_1^p &= CU_1^p - z^{-1/2}CU_1^n \rightarrow i_1^p = CfU_1^p - z^{-1/2}CfU_1^n \\ \Delta q_1^n &= CU_1^n - z^{-1/2}CU_1^p \rightarrow i_1^n = CfU_1^n - z^{-1/2}CfU_1^p \end{aligned} \tag{5.57}$$

The storistor-resistor model of this circuit is shown in Fig. 5.23. In both phases the circuit is symmetrical and contains the parallel connected resistor of the conductance $G=Cf$ and storistor in the form of current source controlled by the voltage from opposite phase.

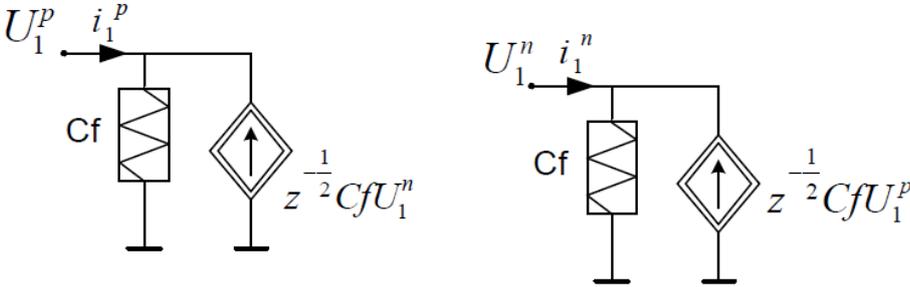


Fig. 5.23. Storistor-resistor model of the grounded capacitor.

Example 5.15

Fig. 5.24 present the floating capacitor included in the circuit in both phases (no switches).

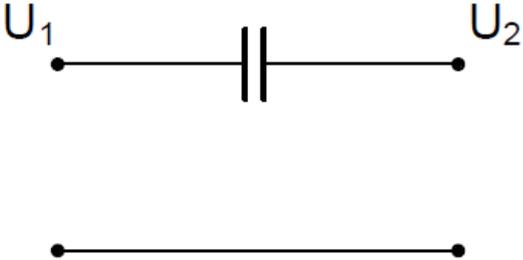


Fig. 5.24 The floating capacitor connection.

The changes of charge of the capacitors in both phases are described by

$$\begin{aligned} \Delta q_1^p &= CU_{12}^p - z^{-1/2}CU_{12}^n \rightarrow i_1^p = CfU_{12}^p - z^{-1/2}CfU_{12}^n \\ \Delta q_1^n &= CU_{12}^n - z^{-1/2}CU_{12}^p \rightarrow i_1^n = CfU_{12}^n - z^{-1/2}CfU_{12}^p \end{aligned} \quad (5.58)$$

The storistor-resistor model of this element is presented in Fig. 5.25. Similarly to the previous example the circuit is symmetrical and contains the parallel connected floating resistor of the conductance $G=Cf$ and storistor in the form of current source controlled by the voltage from opposite phase

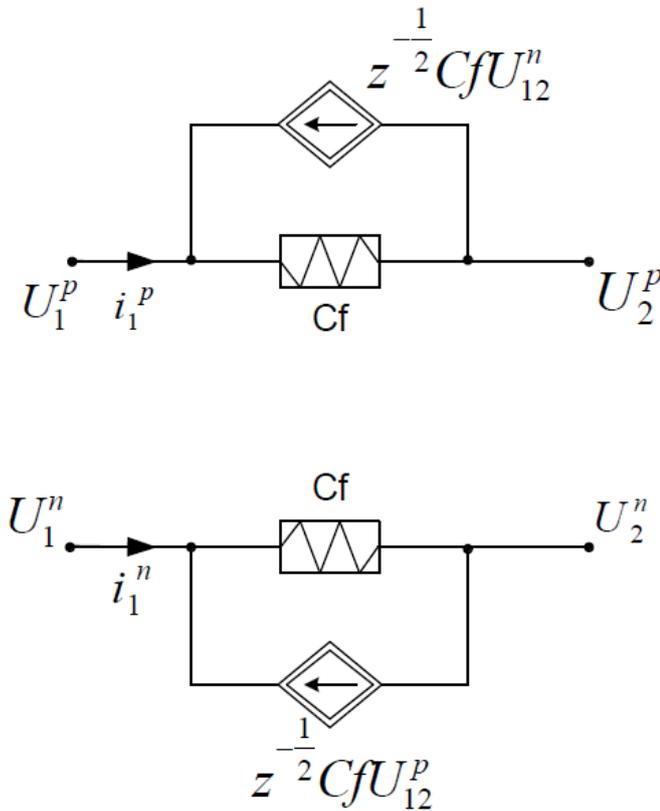


Fig. 5.25 Storistor-resistor model of the floating capacitor.

The SC circuits are commonly used in connection with operational amplifiers. In analysis of such circuits we should apply also the proper model of the amplifiers in both phases. If the amplifiers are included in the circuit without switches they are doubled (one in even and the second in odd phase). Such situation is presented in Fig. 5.26.

If the amplifier is included in the circuit through the switches only in one phase of switching cycle, for example even phase, as shown in Fig. 5.27, its model is included only in this phase. In the opposite phase it is absent.

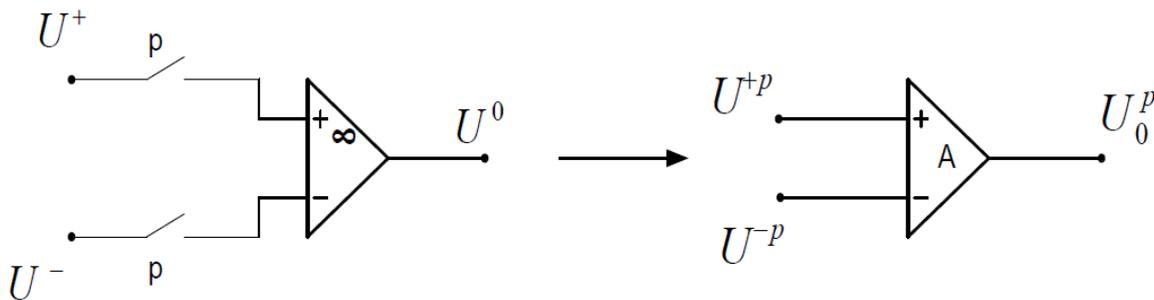


Fig. 5.27 Model of operational amplifier connected to the external circuit only in even phase .

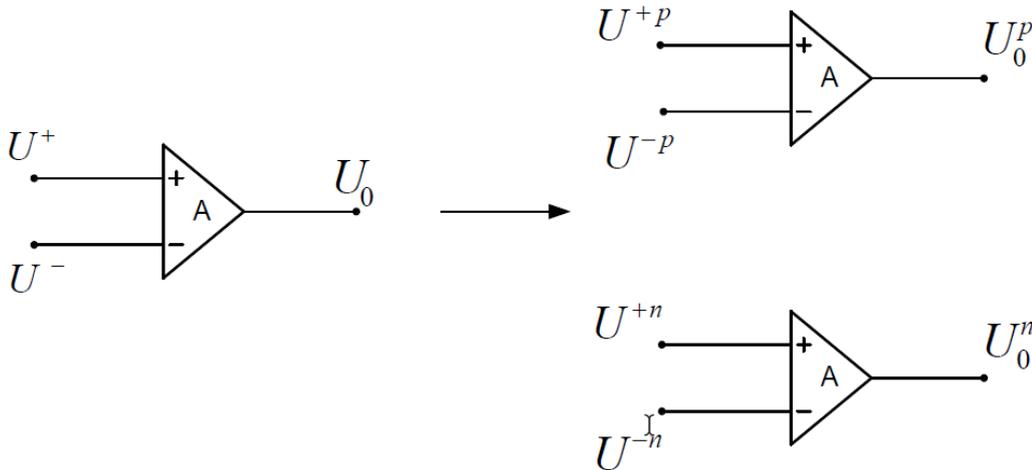


Fig. 5.26. Model of operational amplifier acting in even and odd phase.

5.4.3 Application of SFG in analysis of SC circuits

Analysis of SC circuits is a complex task, since the equivalent resistor-storistor circuit is usually large. In such case good tool of analysis is application of signal flow graphs.

The procedure of analysis may be split into few stages [16]:

- 1) Build the storistor-resistor model for each elements of the SC circuit. This includes the elements with switches, capacitors without switches, operational amplifiers and excitation sources. As a result we get the typical form of electrical circuit containing resistors, storistors, controlled sources and operational amplifiers in both phases of switching cycle.
- 2) In the next step build the signal flow graph using the models discussed in chapter 1.
- 3) In the last step apply the Mason gain formula to calculate the needed transfer functions.

In the case of ideal operational amplifiers (infinite gain, zero output impedance and infinite input impedance) with grounded non-inverting node we suggest to apply the simplified SFG model of the circuit (Fig. 1.6 of chapter 1), which is repeated here in the form shown in Fig.5.28.

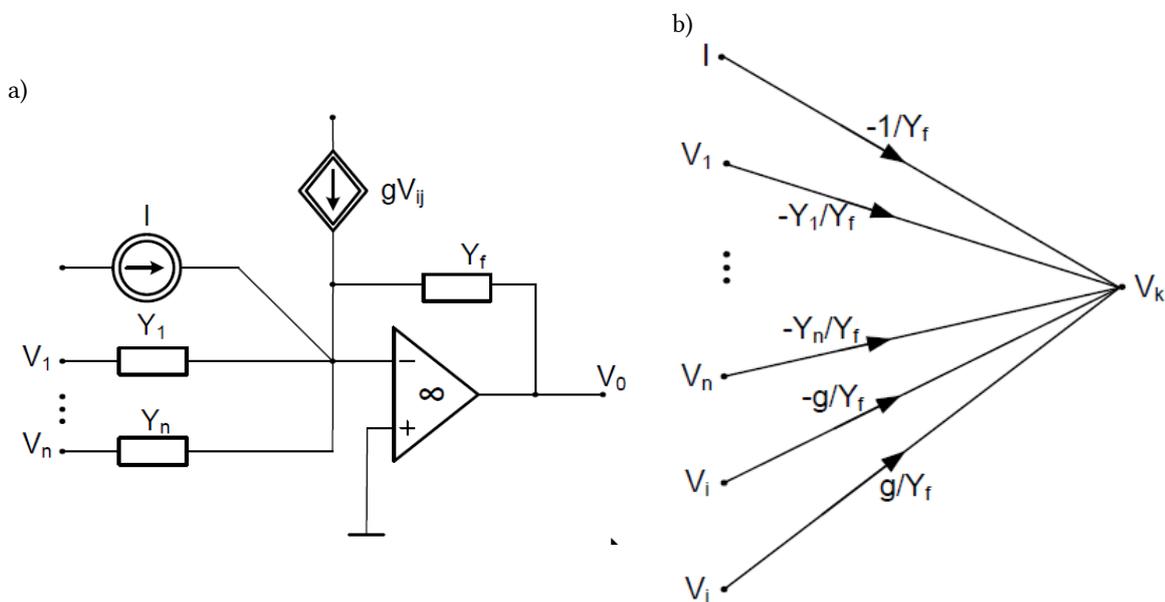


Fig. 5.28. Automatic way of building SFG for the typical connection of operational amplifier with grounded non-inverting input: a) real connection of elements, b) Mason graph corresponding to such connection.

The procedure of analyzing SC circuits with operational amplifiers will be presented on two examples.

Example 5.16

Let us assume the SC implementation of the integrating circuit presented as RC connection in Fig. 5.29a. The resistor should be substituted by proper switched capacitor connection as in Fig. 5.20. The final circuit implementing SC version of integrator is shown in Fig. 5.29b.

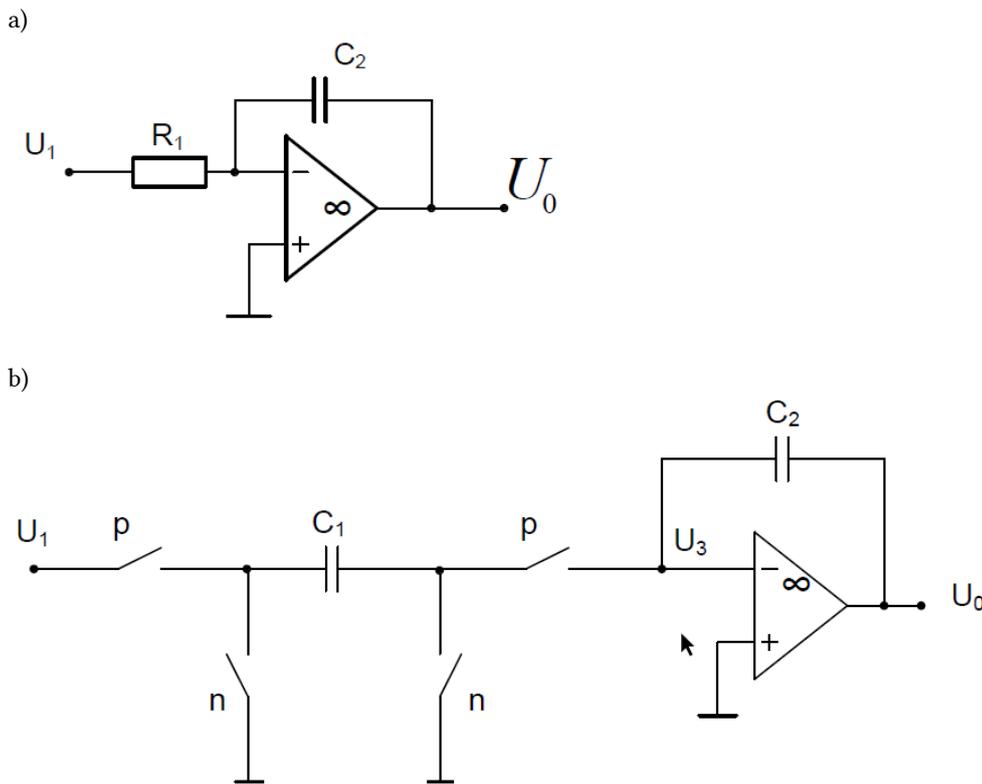
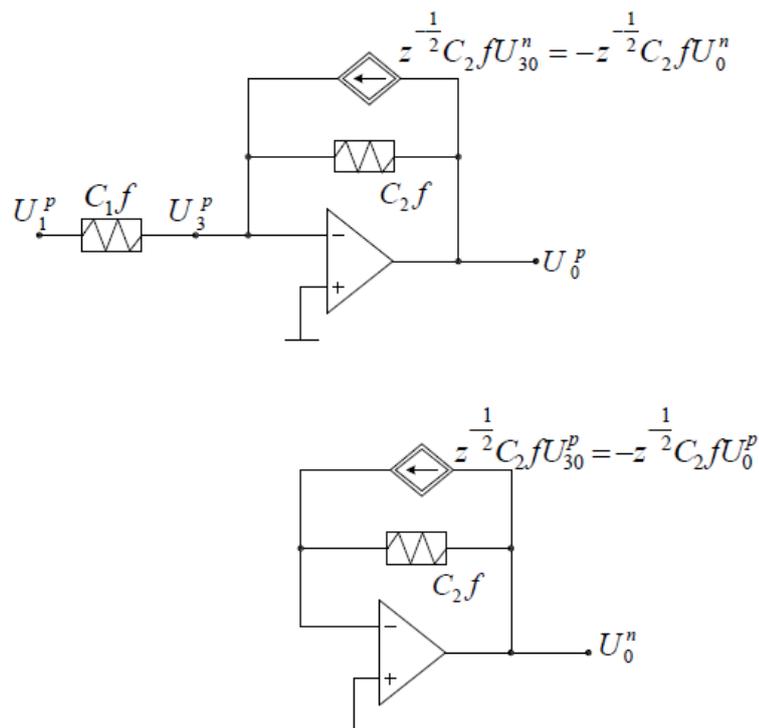


Fig. 5.29. The integrating circuit: a) RC structure, b) SC implementation of the circuit.

Applying the storistor-resistor models of the elements, forming the SC structure, we get its model presented in Fig. 5.30a.

a)



b)

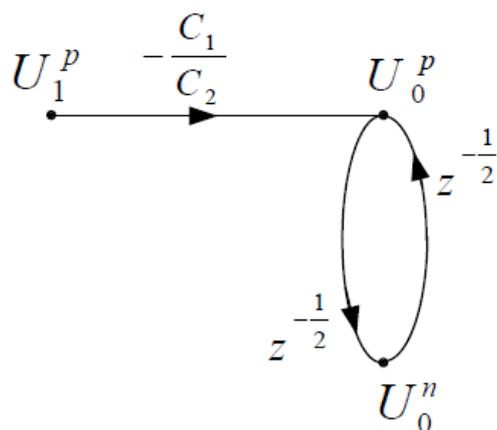


Fig. 5.30. Model of SC integrator: a) storistor-resistor form, b) SFG model.

The signal flow graph corresponding to this storistor-resistor model is shown in Fig. 5.30b. Applying Mason gain formula allows getting two transfer functions $H_{pp}(z)$ and $H_{np}(z)$ as follows

$$H_{pp}(z) = \frac{U_0^p}{U_1^p} = -\frac{C_1}{C_2} \frac{1}{1 - z^{-1}} \quad (5.59)$$

$$H_{np}(z) = \frac{U_0^n}{U_1^p} = -\frac{C_1}{C_2} \frac{z^{-1/2}}{1 - z^{-1}} \quad (5.60)$$

Both forms differ only by the delay term $z^{-1/2}$ corresponding to delay of response by half of the switching cycle. This form of implementation of integrator is known as Lossless Discrete Integrator (LDI).

Example 5.17

The next example shows the structure of the biquadratic filter implemented in SC technology (Fig. 5.31) [22].

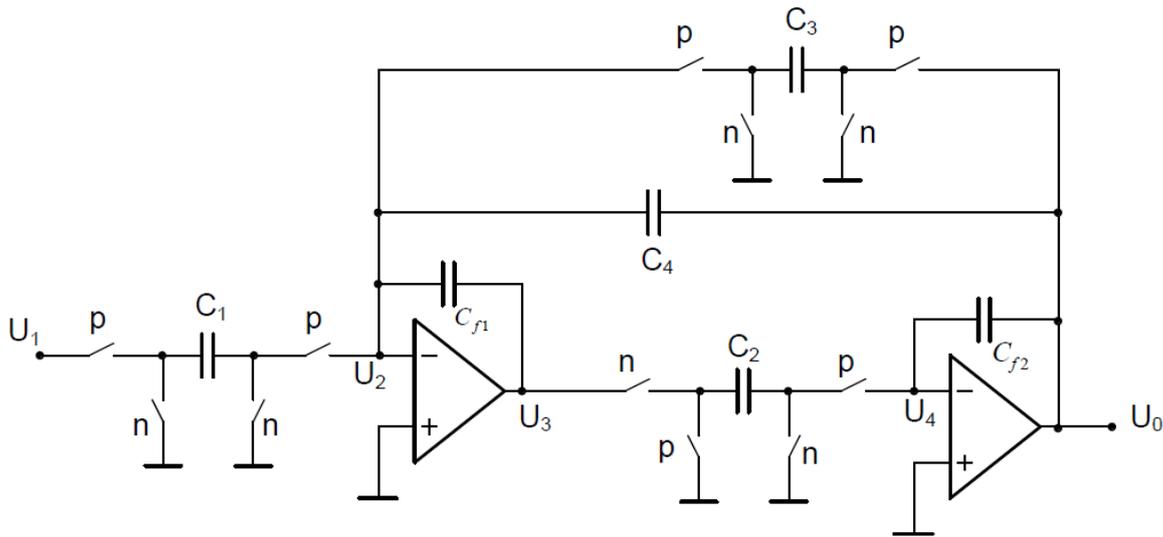


Fig. 5.31. The SC structure of biquadratic filter.

Fig. 5.32 depicts the storistor-resistor model of this circuit. It applies two symmetrical switched capacitor connections (C1, C3), one unsymmetrical four-switch capacitor connection (C2), three non-switched capacitors (Cf1, Cf2, C4) and two non-switched operational amplifiers, treated as ideal.

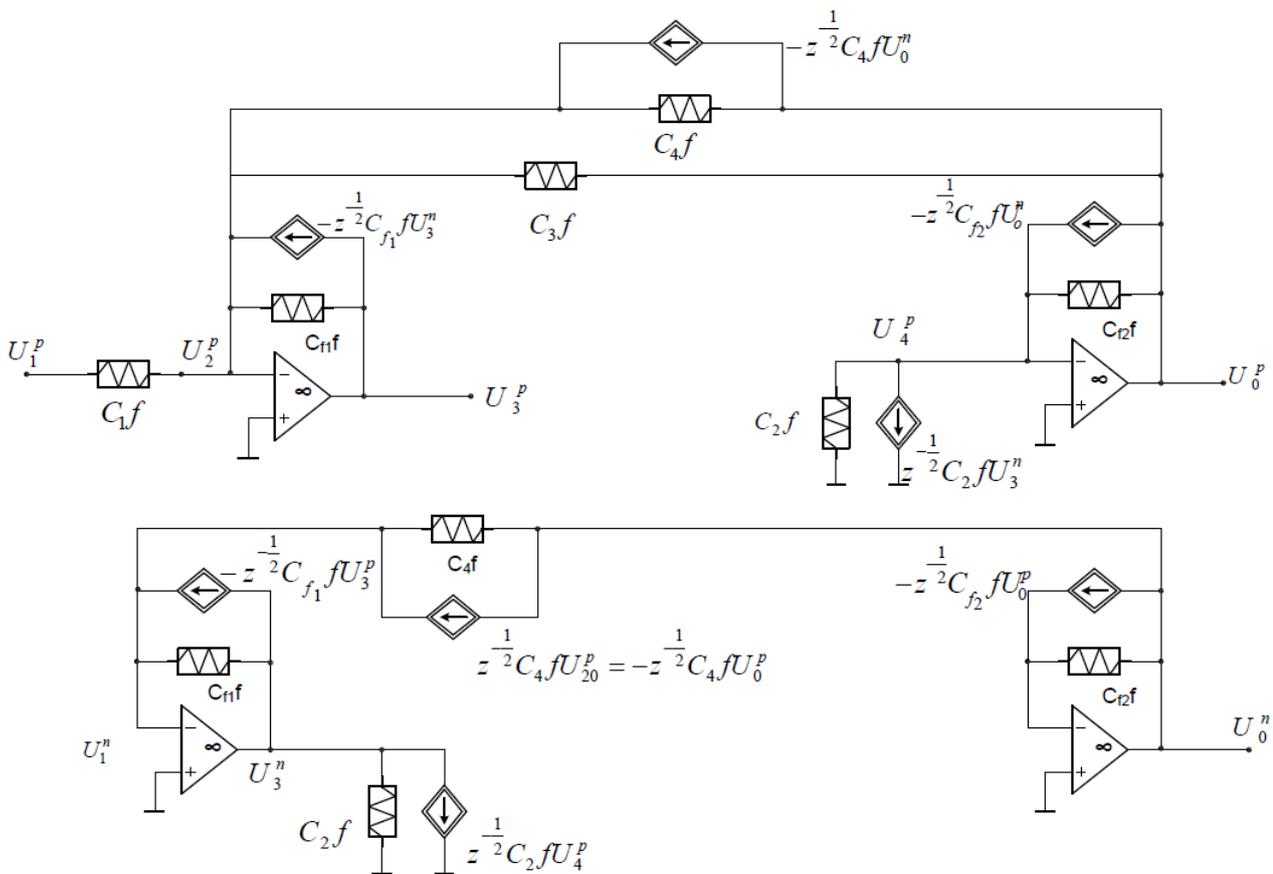


Fig. 5.32. The storistor-resistor model of the biquadratic filter implemented in SC technology.

Applying the procedure of automatic building signal flow graph, the circuit structure of Fig. 5.32 can be easily converted to SFG representation, as shown in Fig. 5.33.

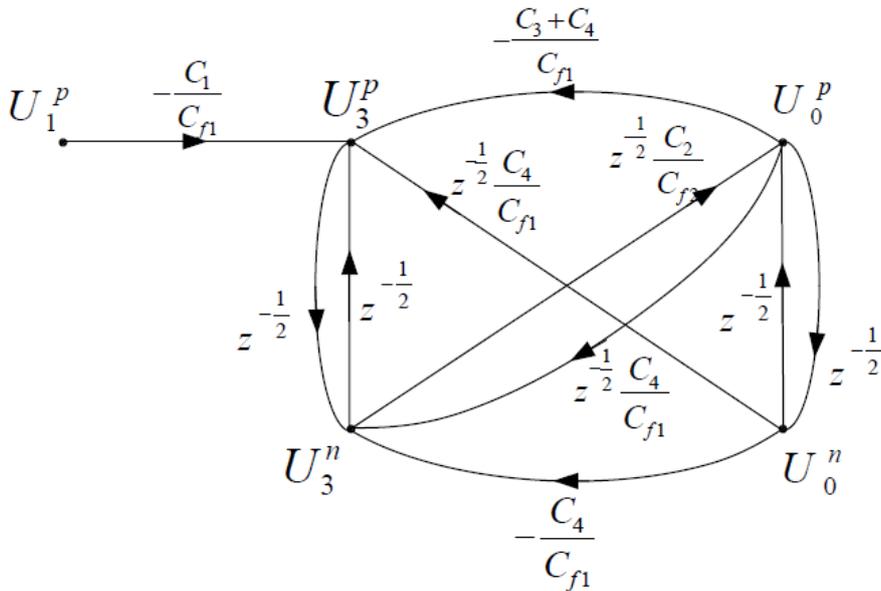


Fig. 5.33. Signal flow graph representing the circuit of Fig. 5.32

Applying Mason formula we obtain two transfer functions of the circuit: $H_{pp}(z)$ and $H_{np}(z)$ in the form

$$H_{pp}(z) = \frac{U_0^p}{U_1^p} = \frac{-\frac{C_1 C_2}{C_{f1} C_{f2}} z^{-1}}{1 + z^{-1} \left[\frac{C_2(C_3 + C_4) - 2C_{f1} C_{f2}}{C_{f1} C_{f2}} \right] + z^{-2}} \quad (5.61)$$

$$H_{np}(z) = \frac{U_0^n}{U_1^p} = \frac{-\frac{C_1 C_2}{C_{f1} C_{f2}} z^{-3/2}}{1 + z^{-1} \left[\frac{C_2(C_3 + C_4) - 2C_{f1} C_{f2}}{C_{f1} C_{f2}} \right] + z^{-2}} \quad (5.62)$$

Observe that these functions are different only by the term $z^{-1/2}$ in the numerator (half of the switching cycle).

5.5 Exercises

1. Apply inverse Z transform to calculate $x(n)$ for
2. a) $X(z) = 2z/(z-0.3)^2$
3. b) $X(z) = 1/(z^2 + 2z + 3)$

Solutions:

Ad a)

There are 2 identical roots $z_1 = z_2 = 0.3$ of the denominator. Therefore, inverse transform is calculated from the expression

$$x(n) = \text{res}(X(z)z^{n-1})_{z=0.3} = \lim_{z \rightarrow 0.3} \frac{d}{dz} [X(z)(z-0.3)^2 z^{n-1}] = 2 \frac{d}{dz} z^n = 2n z^{n-1} = 2n(0.3)^{n-1} \cdot 1(n)$$

Ad b)

This function has two single roots $z_1 = -1$ and $z_2 = -2$ of the denominator. Therefore, the inverse transform is calculated from the expression

$$x(n) = \operatorname{res}(X(z)z^{n-1})|_{z=-1} + \operatorname{res}(X(z)z^{n-1})|_{z=-2} = \frac{z^{n-1}}{(z+2)}|_{z=-1} + \frac{z^{n-1}}{(z+1)}|_{z=-2} = (-(-1)^n - 0.5(-2)^n) \cdot 1(n)$$

1. Determine transfer function $H(z)=Y(z)/U(z)$ for discrete system described by the following difference equation

$$y(n)=2y(n-1)+3y(n-2)-0.5y(n-3)+5x(n)-0.7x(n-1)$$

Solution:

Applying Z-transform to both sides of equation we get

$$Y(z)=2z^{-1}Y(z)+3z^{-2}Y(z)-0.5z^{-3}Y(z)+5X(z)-0.7z^{-1}X(z)$$

Hence

$$H(z) = \frac{Y(z)}{X(z)} = \frac{5 - 0.7z^{-1}}{1 - 2z^{-1} - 3z^{-2} + 0.5z^{-3}}$$

1. Transform $H(s)=1/(s^2+2s+5)$ into $H(z)$ using the bilinear transformation

Solution:

In bilinear transformation we apply the following mapping

$$s = 2f_s \frac{1 - z^{-1}}{1 + z^{-1}}$$

First, we have to estimate value of sampling rate f_s . Assume the maximum frequency f_{max} of the filter, at which the magnitude characteristic drops below 0.01. Hence , we get

$$H(\omega) = \frac{1}{(5 - \omega^2) + j2\omega}$$

$$|H(\omega)| = \frac{1}{\sqrt{(5 - \omega^2)^2 + 4\omega^2}} \leq 0.01$$

Solving this equation with respect to ω we get $\omega_{max}=103$. It means, that $f_{max}= \omega_{max}/2\pi=16.4\text{Hz}$. Sampling rate f_s should satisfy $f_s>2f_{max}$. Let us choose $f_s=50$. Hence

$$s = 100 \frac{1 - z^{-1}}{1 + z^{-1}}$$

Supplying it into $H(s)$ we obtain transfer function of discrete system in the form

$$H(z) = \frac{10^{-4} (0.247 + 0.495z^{-1} + 0.247z^{-2})}{1 - 1.982z^{-1} + 0.980z^{-2}}$$

1. Draw storistor-resistor model for the SC circuit shown below

Solution:

The storistor-resistor model is presented below

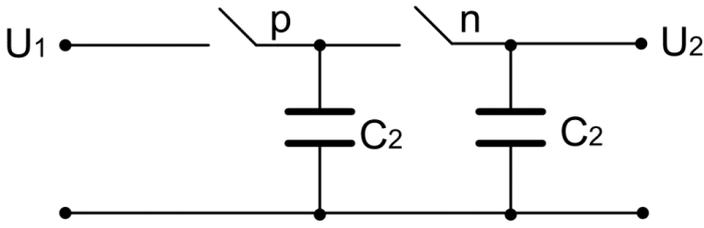


Fig. 5.34a Switched capacitor circuit structure

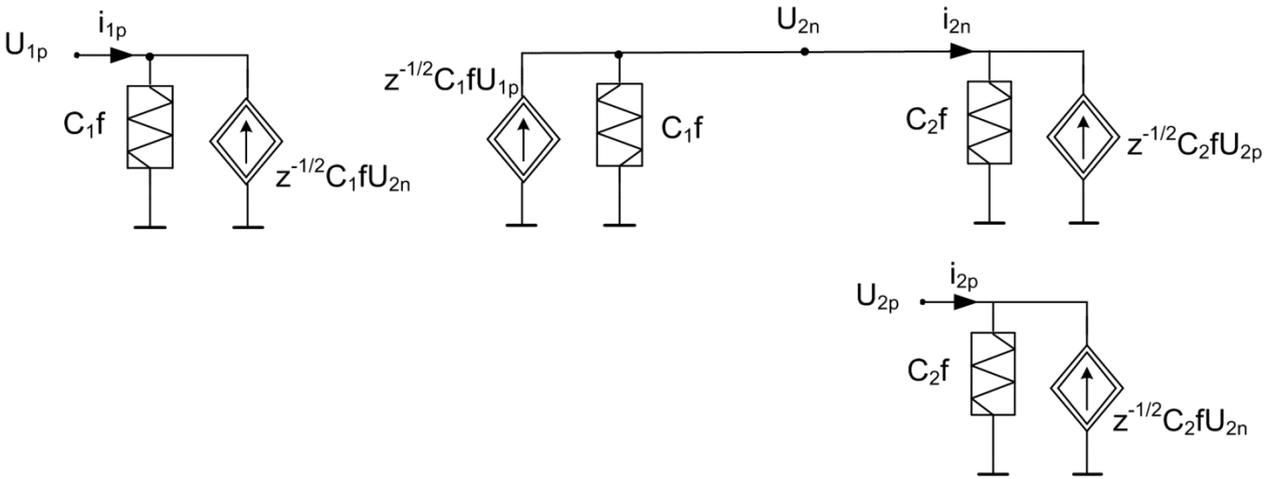


Fig. 5.34b The storistor-resistor model of the SC circuit

1. Draw storistor-resistor model for the SC circuit with operational amplifier shown in Fig. 5.35a.

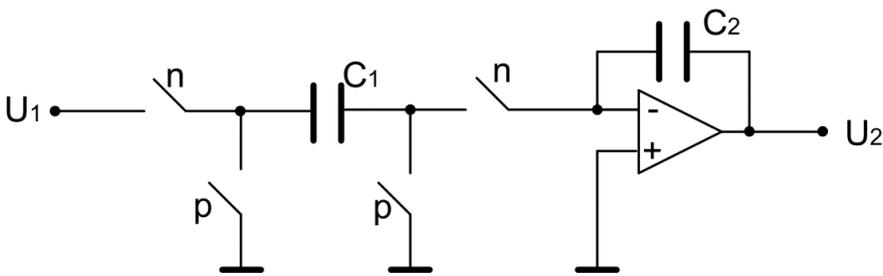


Fig. 5.35a Switched capacitor circuit structure

Solution:

The storistor-resistor model of the SC circuit is presented below

Basic definitions

Backward Euler method – is the most basic numerical implicit method for the solution of ordinary differential equations. It implements the differentiation of signal by applying finite difference in step back.

BIBO stability – short name of bounded-input, bounded-output. It is a form of stability for linear systems that take inputs. The system is BIBO stable, if the output is bounded for every input to the system that is bounded.

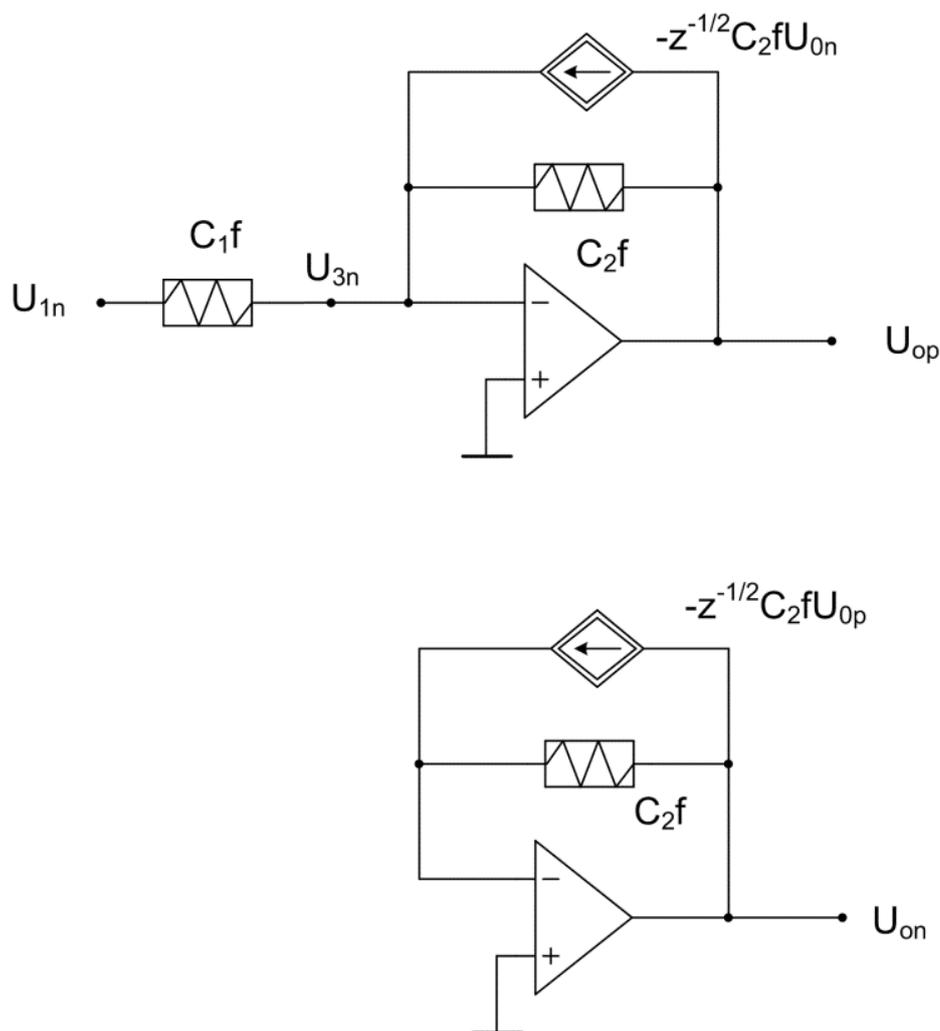


Fig. 5.35b The stator-resistor model of the SC circuit from Fig. 5.35a

Bilinear method – the first order transformation of continuous-time system to discrete-time representation, and vice versa. It is based on trapezoidal rule of integration.

Difference equation – the equation defining the actual response of discrete system on the basis of its past values determined in discrete time instants.

Convolution – the mathematical operation on two signals x and y to produce a third signal z as the integral of the product of the two signals after one is reversed and shifted. The convolution of x and y is written x^*y , using an **asterisk** or star.

Discrete system – the type of system, in which information of signals is accessible only at discrete time points $t_k = kT_s$, where T_s is a sampling period and k the time index. It is usually contrasted with continuous systems (called analog systems), in which the signal information is available for any point of time.

Even phase – The even phase of clock in which the capacitor samples the analog voltage through the closed switch.

FIR system – the discrete system whose **impulse response** (or response to any finite length input) is of finite duration, because it settles to zero in finite time.

Forward Euler method – the **numerical explicit method for the solution of ordinary differential equations**. It implements the differentiation of signal by applying finite difference in step forward.

IIR system – the discrete system whose **impulse response** (or response to any finite length input) is of infinite duration. Such system has internal feedback that may continue to respond indefinitely (usually decaying with time).

Invariance of impulse response – method of transforming analog system into discrete system description. It assumes that impulse response in sampling points of both (analog and discrete) systems should be equal.

Inverse Z-transform – conversion of complex [frequency-domain](#) z-representation of signal back to the original discrete time signal description.

Odd phase – The odd (alternative to even) phase of clock in which the capacitor samples the analog voltage through the closed switch.

State space representation – mathematical model of a physical system as a set of input, output and state variables related by first-order [differential equations](#) (analog systems) or [difference equations](#) (discrete systems). State variables are internal variables whose values evolve through time and allow determining uniquely any output variables.

Storistance – the controlling coefficient of the storistor represented usually by the admittance type description.

Storistor – the memory element representing some memory from the previous phase in switched capacitor circuit. It is usually presented a form of voltage controlled current source.

Storistor-resistor model – the conventional circuit model of switched capacitor circuit operating with equivalent resistors and storistors described by their admittance functions representing the average voltage and current relations within the switching period.

Switched capacitor – [electronic circuit](#) element used for [discrete-time signal processing](#). It works by moving charges into and out of [capacitors](#) when [switches](#) are opened and closed.

Unity delay – the delay of signal by one sampling period. When placed in an iterating subsystem, it holds and delays its input in one iteration. This block is equivalent to the z^{-1} discrete-time operator.

Z-transform – conversion of a [discrete-time signal](#) (a [sequence](#) of [real](#) or [complex numbers](#)) into a complex [frequency-domain](#) representation. It is a discrete-time equivalent of the [Laplace transform](#).

6. Nonlinear elements

To the important class belong nonlinear circuits, the circuits containing at least one nonlinear element [3]. In contrast to linear the nonlinear element is characterized by the nonlinear relation between the input variable $x(t)$ and output response $y(t)$. The response depends not only on the input signal, but also on the internal parameters representing the nonlinearity with respect to the input signal, i.e.,

$y(t) = f(x(t), p_1, p_2, \dots, p_n) = 0$, where p_1, p_2, \dots, p_n are the internal parameters of the element. If these parameters are independent on time the nonlinear element is stationary. The nonlinear circuit does not obey the superposition rule. It means, that in general case

$$y(x_1 + x_2) \neq y(x_1) + y(x_2) \quad (6.1)$$

Hence, in analysis of nonlinear circuit we have to take into account simultaneously all excitations existing in the circuit, even, when they represent different characters (for example DC, sinusoidal, exponential, etc.). Moreover, the solution of most practical nonlinear circuits seldom exists in analytic form. Consequently, the solution of the most practical nonlinear circuits must be obtained either by graphical or numerical methods.

6.1. Basic notions of nonlinear elements

The description of the nonlinear element may involve different variables associated with the element: current i , voltage u , charge q of capacitor or flux linkage ψ in magnetic coil. These variables satisfy the basic definitions

$$\begin{aligned} u &= \frac{d\psi}{dt} \\ i &= \frac{dq}{dt} \end{aligned} \quad (6.2)$$

and represent the nonlinear relations. In general case we can recognize four types of nonlinear elements [3,16]

1. Nonlinear resistor characterized by the current-voltage relation (u, i)
2. Nonlinear coil characterized by the flux-current relations (ψ, i)
3. Nonlinear capacitor characterized by the charge-voltage relation (q, u)
4. Nonlinear memristor defined by the general relations charge-flux (ψ, q)

Let us assume the notation x for independent variable and y for dependent one. The above general implicit descriptions of the elements might be converted to explicit form (if exists) by expressing one variable as the function of the second one. In such case three types of elements can be recognized

- Element controlled by x , described explicitly by $y=f(x)$

- Element controlled by y , described explicitly by $x=g(y)$
- Element of monotonic characteristics, having both possible descriptions: $y=f(x)$ and $x=g(y)$.

Each nonlinear characteristic can be associated with two types of parameters: **static** and **dynamic**. Assume for simplicity element described by $y=f(x)$. In the operating point x_0 we can define [3,16]

- Static parameter $\tau(x_0)$

$$\tau(x_0) = \frac{y(x_0)}{x_0} \quad (6.3)$$

- Dynamic parameter $D(x_0)$

$$D(x_0) = \left. \frac{dy}{dx} \right|_{x=x_0} = \left. \frac{df(x)}{dx} \right|_{x=x_0} \quad (6.4)$$

In the case of linear elements these parameters are equal. However, this is not the case for nonlinear elements.

Example 6.1

Let us consider the nonlinear element of the input-output characteristic $y=f(x)$ shown in Fig.6.1. Applying definitions (6.3) and (6.4) we get the static and dynamic characteristics shown in Fig. 6.1. They differ significantly. From the practical point of view only dynamic characteristic is important in electrical engineering.

The nonlinear elements may be **stationary** or **non-stationary**. [3]. The nonlinear element is stationary, if its parameters do not depend on time and only on input variable x . It means that the dynamic parameter depends only on the input variable, that is $D=D(x)$. Otherwise the element is non-stationary, for which $D=D(x,t)$.

An example of nonlinear element is resistor described by the function $i = 5u^3 + 3u$. The dynamic parameter represents the conductance G , which is expressed by $G = di/du = 15u^2 + 3 = G(u)$. In the case, when the nonlinear description of elements includes time in explicit way, for example $i = 5u^3t + 3u$ the element is non-stationary and its dynamic parameter is a function of time, $G = di/du = 15u^2t + 3 = G(u, t)$.

The nonlinear elements might be **passive** or **active**. Passive element does not generate the energy at any point of time and active element is a source of energy, at least in some region of operating points. From the mathematical point of view the passivity is strictly connected with the following condition [3,13,16]

$$E = E(t) - E(t_0) = \int_{t_0}^{\infty} u(t)i(t)dt \quad (6.5)$$

The element is passive, if for any point of time we have $E \geq 0$, and active, if in some period of time we deal with a situation, that $E < 0$. If $E = 0$, the element is passive and **lossless**. The examples of passive and active element characteristics are presented in Fig. 6.2.

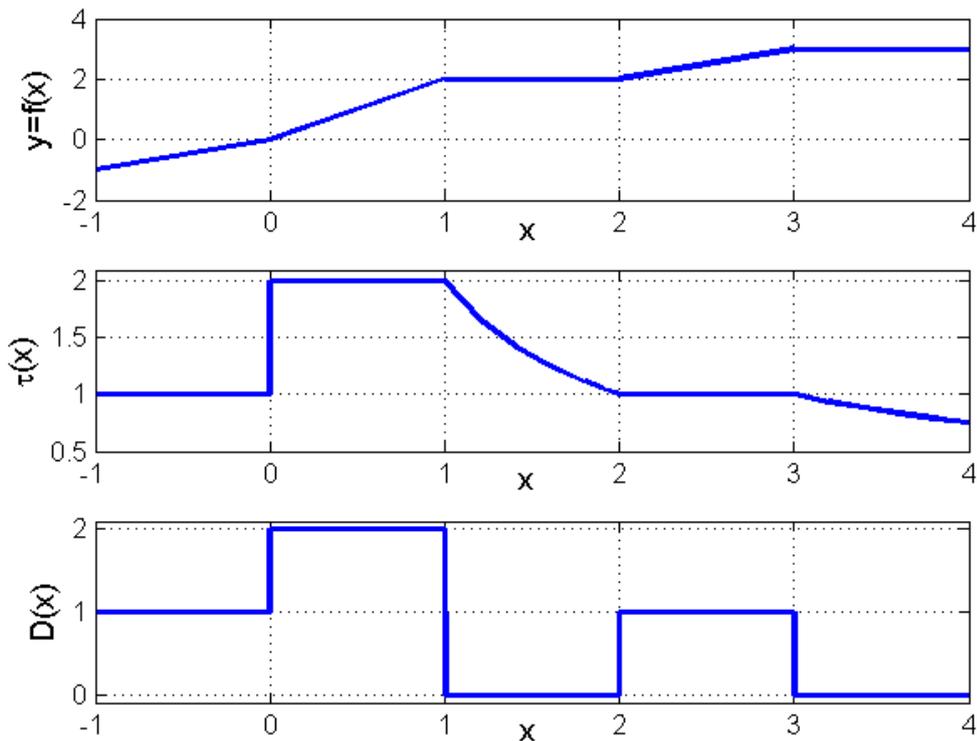


Fig. 6.1. Graphical interpretation of static and dynamic parameters of nonlinear element. Upper graph shows the nonlinear characteristic of the element, middle – the static parameter changes and bottom – the dynamic parameters.

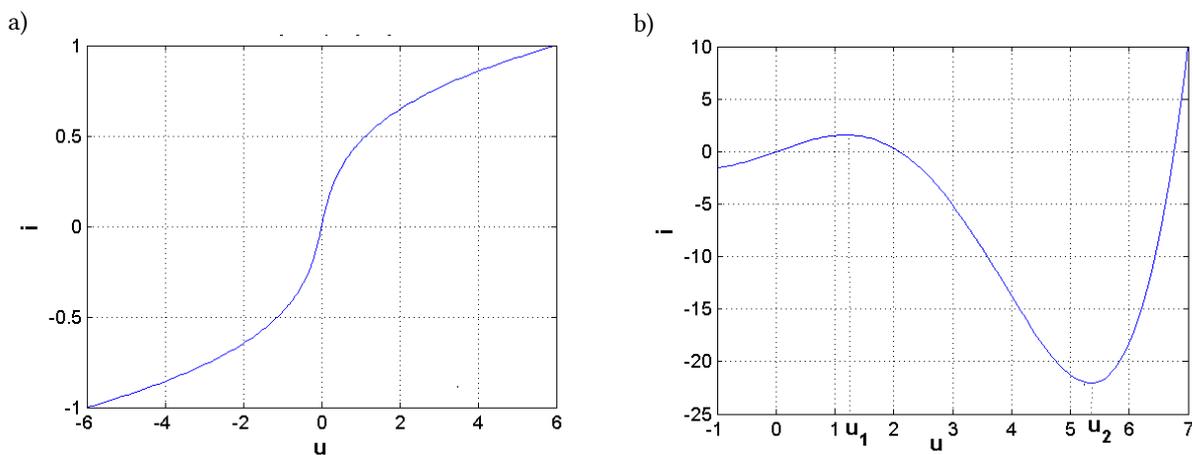


Fig. 6.2. The example of current-voltage characteristic of a) passive, b) active resistive element.

In the case of passive element the derivative of characteristic for all values of voltage is positive (the dynamic conductance is positive). The active element of Fig. 6.2b shows negative values of derivative in the region extending from u_1 to u_2 (the negative dynamic conductance).

6.2. Nonlinear resistive elements

In this class we can recognize 2-terminal elements, called simply nonlinear resistors, and multi-terminal (for

example 3-terminal) elements. The typical symbols used in notation of nonlinear resistors are presented in Fig. 6.3 [3,16]. The resistive element can be completely characterized by the voltage-current relations. The directions of current and voltage are opposite.

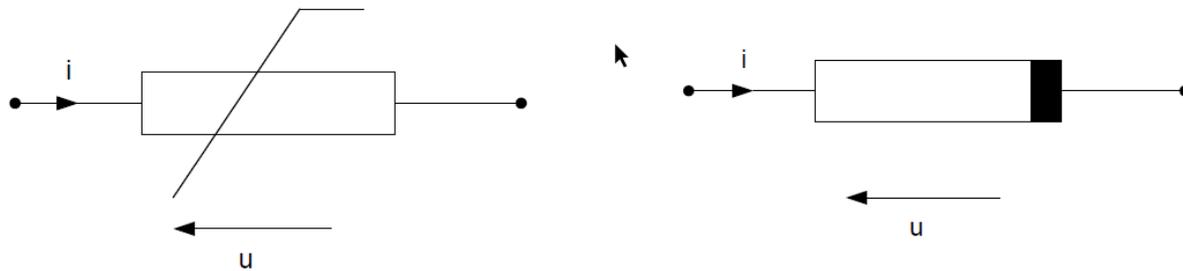


Fig. 6.3. Two symbols of nonlinear resistor used in practice.

The general description of resistor, which always exists, is given in an implicit form $f(u_R, i_R) = 0$. If the voltage-current characteristic is unambiguous with respect to either current or voltage we can recognize three types of elements:

- Resistor controlled by the voltage, description $i_R = g(u_R)$
- Resistor controlled by the current, description $u_R = f(i_R)$
- Monotonic resistor, possessing both types of description $i_R = g(u_R)$ and $u_R = f(i_R)$, while $u_R = g^{-1}(i_R)$.

Depending on the type of characteristic we can define dynamic parameter as:

- Resistance $R(i_R) = \frac{du_R}{di_R}$ for current controlled element
- Conductance $G(u_R) = \frac{di_R}{du_R}$ for voltage controlled element

In contrast to linear element the dynamic resistance of the nonlinear element is the function of the controlling variable (either current or voltage). In the case of monotonic characteristic both descriptions (resistance and conductance) are determined in a unique way. The resistor is passive if for any time point the following inequality condition holds

$$E_R = \int_{-\infty}^{\infty} u_R i_R dt \geq 0 \quad (6.6)$$

The passivity condition is satisfied if the dynamic parameter is positive in the whole range of input signal values.

Among nonlinear resistive elements the important class are the multi-terminal elements. In such case the voltages and currents of terminals form the vectors, denoted as following.

$$\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The general description of the n -terminal element can be described in an implicit form $\mathbf{f}(\mathbf{u}, \mathbf{i}) = \mathbf{0}$. In most cases this implicit vector form can be expanded into few explicit relations, characterizing one variable as the function of the others.

6.2.1. Examples of nonlinear 2-terminal nonlinear resistors

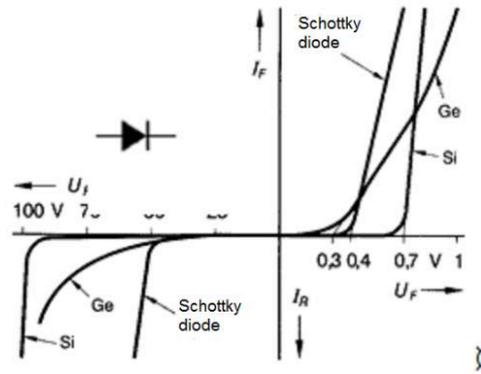
The typical examples of nonlinear 2-terminal resistors are the electronic elements: diode, Zener diode, tunnel diode, diac, varistor, thermistor, etc. Table 6.1 presents symbols and current-voltage characteristics of some chosen elements [16].

Table 6.1 Examples of nonlinear 2-terminal resistive elements.

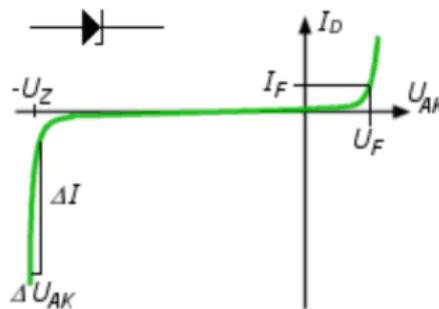
Name

Symbol and characteristics of elements

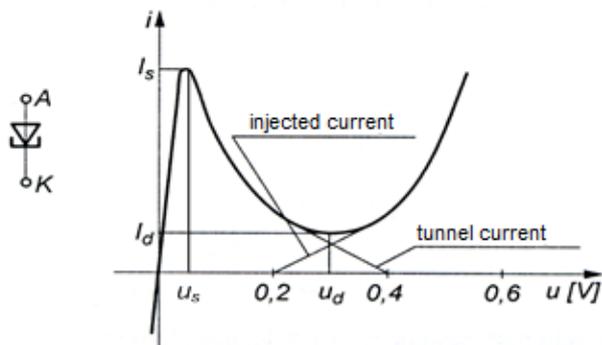
Diode



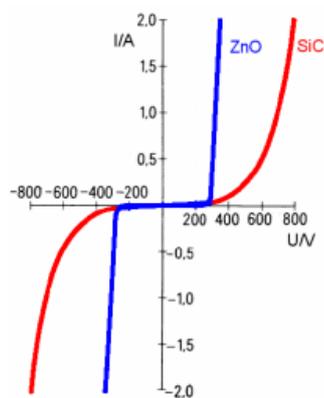
Zener diode



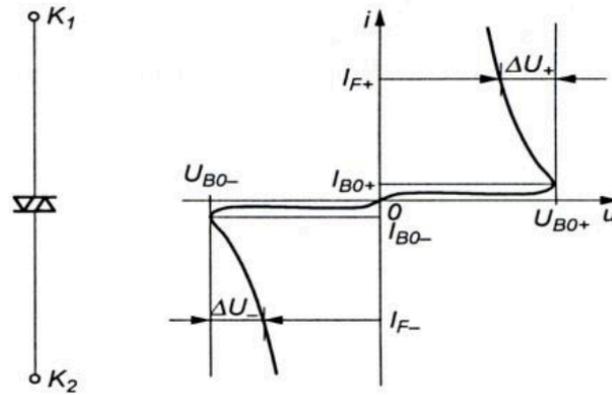
Tunnel diode



Varistor



Diac



The diode construction is based on p-n junction. This junction can be activated in forward or reverse direction. A p-n junction diode allows electric charges to flow in one direction, but not in the opposite one. The negative charges (electrons) can easily flow through the junction from n to p but not from p to n, and the reverse is true for holes. When the p-n junction is forward-biased, electric charge flows freely due to reduced resistance of the p-n junction (diode represents very small resistance). When the p-n junction is reverse-biased, however, the junction barrier (and therefore resistance) becomes greater and charge flow is minimal (diode resembles open circuit).

Signal diodes are produced for small currents (typically below 1A) and low voltages (typically below 50V). Power diodes are built for very high currents and voltages. Both have the rectification property, allowing the current flow only in one conducting direction. The mathematical description of current-voltage (i_d , u_d) relation of diodes is usually presented in the form (valid for small frequency)

$$i_d = I_0 \left(e^{\frac{u_d}{Vt}} - 1 \right) \quad (6.7)$$

The variable Vt represents the thermal voltage (typical $Vt \approx 26\text{mV}$ for silicon) and I_0 is the saturation current or scale current of the diode. Analyzing this relations we find that above some thresholding value of voltage the current grows rapidly. The value of this threshold depends on material, from which the diode is made. For silicon it is about 0.7V, for germanium 0.4V, for Schottky diode 0.3. In many higher voltage applications diode can be treated either as short circuit (in the conducting mode) or open circuit (in non-conducting mode).

A Zener diode is a particular type of diode that, unlike a normal one, allows current to flow not only from its anode to its cathode, but also in the reverse direction, when the Zener voltage is reached. Zener diodes **break down** with a reverse voltage but the voltage and sharpness of the knee are well defined. Zener diodes can reliably operate in this region, providing the stable operation with a stabilized voltage.

A tunnel diode, called also Esaki diode, is a type of semiconductor diode that has **negative resistance** in some range of voltages due to the **quantum mechanical** effect called **tunneling**. In the diode characteristic presented in Table 6.1 this range extends from U_s to U_d . The tunnel diodes show great promise as an oscillator and high-frequency threshold (trigger) device.

A varistor is an **electronic component** with an electrical **resistance** that varies with the applied voltage. It represents a voltage-dependent resistor (VDR), which may be described by

$$U = a \cdot I^\beta \quad (6.8)$$

Its characteristic is **nonlinear**, and to some degree similar to that of a diode. In contrast to a diode, however, it has the same characteristic for both directions of traversing current. At low voltage it has a high electrical resistance which decreases as the voltage is raised.

The diac (acronym of “diode for alternating current”) is a diode that conducts **electrical current** only after its breakover **voltage**, V_{B0} , has been reached momentarily. When breakdown occurs, the diode enters a region

of **negative dynamic resistance**, leading to a decrease in the voltage drop across the diode and, usually, a sharp increase in current through the diode. The diode remains in conduction until the current through it drops below a value characteristic for the device, called the holding current. Below this value, the diode switches back to its high-resistance, non-conducting state. This behavior is bidirectional, meaning typically the same for both directions of current.

6.2.2. Examples of 3-terminal nonlinear elements

The 3-terminal elements represent very important class of nonlinear resistors (in a low frequency applications). To this class belong bipolar transistors, field-effect transistors, thyristors and triacs. Table 6.2 presents some chosen 3-terminal elements and their current-voltage characteristics [16].

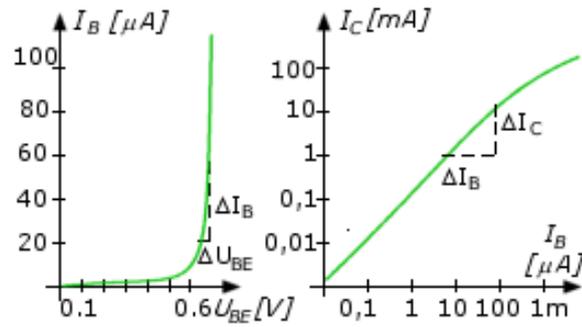
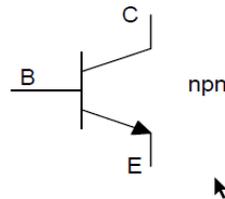
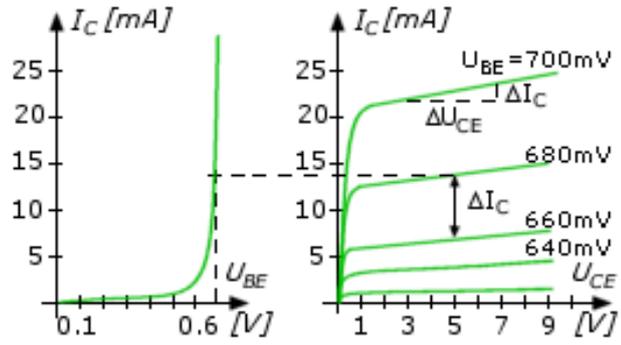
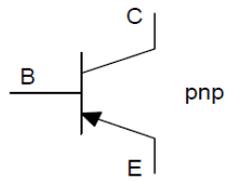
Table 6.2 Examples of chosen 3-terminal nonlinear elements

Name

Symbol

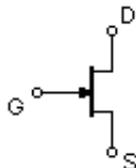
Characteristics of elements

Bipolar transistor

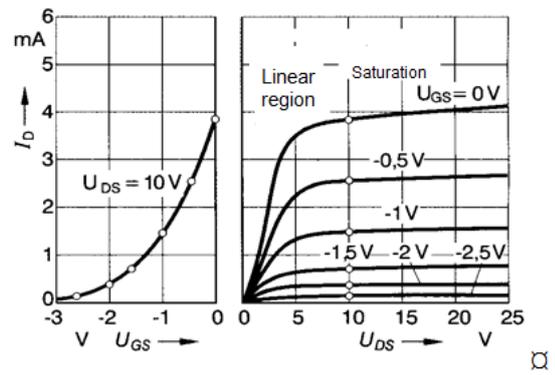
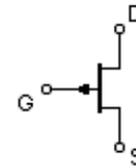


Field-effect transistor

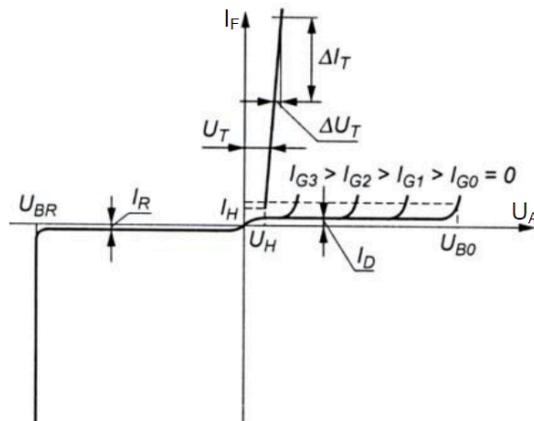
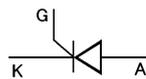
n-channel



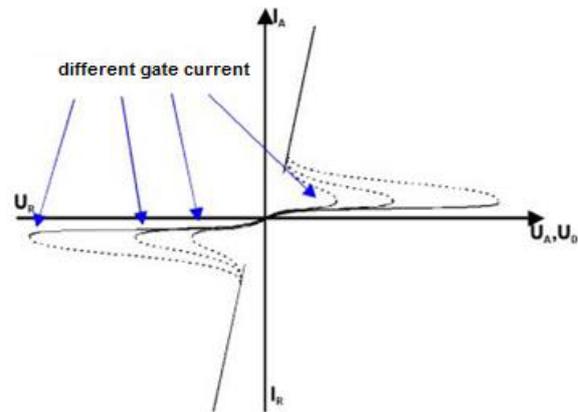
p-channel



Thyristor



Triac



A bipolar junction transistor is a type of [transistor](#) that uses both [electron](#) and [hole](#) charge carriers. In contrast, unipolar transistors, such as [field-effect transistors](#), only use one kind of charge carrier. Bipolar transistors use two junctions between two [semiconductor](#) types, n-type and p-type and are manufactured as NPN and PNP. They are available as individual components, or fabricated in [integrated circuits](#), very often in large numbers. Bipolar transistors have three terminals called: collector, emitter and base.

The collector–emitter current can be viewed as being controlled by the base–emitter current (current control), or by the base–emitter voltage (voltage control). These views are related by the current–voltage relation of the base–emitter junction, which is the usual exponential current–voltage curve of a [p–n junction](#) (diode).

Bipolar transistor can be characterized by different parameters defined on the basis of input and output characteristics. For example, the input dynamic resistance r_{BE} is defined from the characteristic $U_{BE}=f(I_B)$

$$r_{BE} = \left. \frac{\partial U_{BE}}{\partial I_B} \right|_{U_{CE}=\text{const}} \quad (6.9)$$

From the output characteristic $I_C = f(U_{CE})$ assuming $U_{BE}=\text{const}$ or $I_B = \text{const}$ we can define the output resistance r_{CE}

$$r_{CE} = \left. \frac{\partial U_{CE}}{\partial I_C} \right|_{I_B=\text{const}} \quad (6.10)$$

From the characteristic $I_C = f(I_B)$ assuming $U_{CE} = \text{const}$ we obtain the current gain characteristics

$$\beta = \left. \frac{\partial I_C}{\partial I_B} \right|_{U_{CE}=\text{const}} \quad (6.11)$$

We can recognize the [common-emitter](#) current gain represented by βF and approximated as the ratio of the DC collector current to the DC base current in forward-active region $\beta F=I_C/I_B$. It is typically greater than 50 for small-signal transistors, but can be smaller in transistors for high-power applications. Another parameter is the [common-base](#) current gain, αF , treated as the gain of current from emitter to collector in the forward-active region, $\alpha F=I_C/I_E$. This ratio usually has a value close to unity; between 0.980 and 0.998. Both gains are related to each other as $\beta F= \alpha F / (1- \alpha F)$.

The field-effect transistor (FET), known also as unipolar transistor, is a [transistor](#) that uses an [electric field](#) to control the electrical behavior of the device. Many different implementations of field effect transistors exist (JFET, MOSFET, GASFET, etc.). The FET's three terminals are:

1. Source (S), through which the carriers enter the channel. The current entering the channel at S is designated by I_S .
2. Drain (D), through which the carriers leave the channel. The current entering the channel at D is designated by I_D . Drain-to-source voltage is U_{DS} .
3. Gate (G), the terminal that modulates the channel conductivity. By applying voltage to G, one can control the drain current I_D .

Field effect transistors generally display very **high input impedance** at low frequencies. The **conductivity** between the drain and source terminals is controlled by an electric field in the device, which is generated by the voltage difference between the body and the gate of the device. The characteristics curves show three different regions of operation for a MOSFET and these are:

- Cut off region – region where the U_{GS} is less than threshold voltage U_p . For MOS to be in cut-off region, the necessary conditions are $0 < U_{GS} < U_p$ for NMOS and $0 > U_{GS} > U_p$ for PMOS (threshold voltage of PMOS is negative). The magnitude of current flowing through MOS in cut-off region is negligible as the channel is not present.
- Linear or non-saturation region – for NMOS, as gate voltage increases beyond threshold voltage, channel is formed between source and drain terminals. When the voltage difference between source and drain exists, the current will flow. The magnitude of current increases linearly with increasing drain voltage till a particular drain voltage determined by the relations $U_{GS} \geq U_p$ and $U_{DS} < U_{GS} - U_p$. The current is represented as a linear function of U_{GS} and U_{DS} voltages. The linear region voltage-current relation is given as follows:

$$I_D = \frac{I_{DSS}}{U_p^2} [2(U_{GS} - U_p)U_{DS} - U_{DS}^2] \quad (6.11)$$

Similar relations are for PMOS.

- Saturation Region – at a certain gate and source voltage, there is a particular level of voltage U_{DS} , beyond which, increasing drain voltage has no effect on current. The conditions are given as: $U_{GS} \geq U_p$ and $U_{DS} > U_{GS} - U_p$. The current now, is a function only of gate and source voltages:

$$I_D = I_{DSS} \left(1 - \frac{U_{GS}}{U_p}\right)^2 \quad (6.12)$$

Field-effect transistors have found very wide applications in electronics. One specific one, interesting from theoretical point of view is constant-current diode. The symbol of it and structure of implementation are shown in Fig. 6.4.

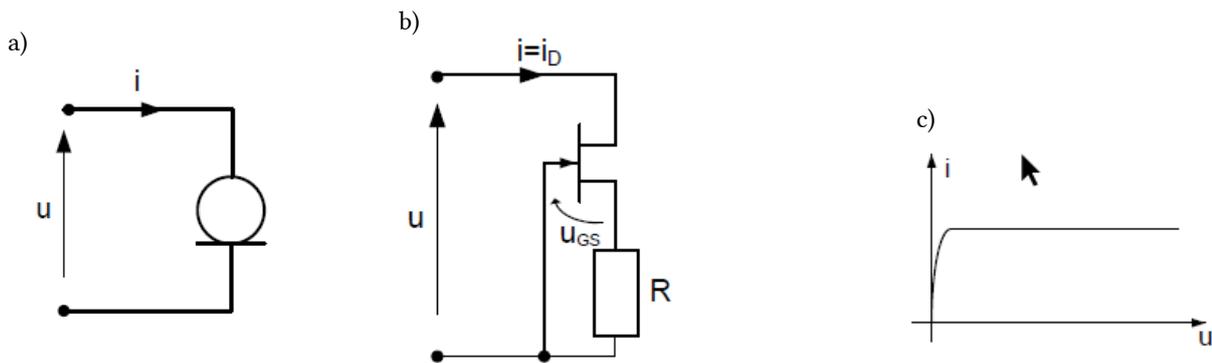


Fig. 6.4. Constant-current diode: a) symbol, b) circuit configuration based om FET, c) current-voltage characteristic.

The operating point of transistor is in the saturation region. The value of current needs to solve the following nonlinear equation ($U_{GS} = -RI_D$)

$$I_D = I_{DSS} \left(1 - \frac{|U_{GS}|}{U_p} \right)^2 = I_{DSS} \left(1 - \frac{RI_D}{U_p} \right)^2 \quad (6.14)$$

This value depends on the parameters of FET and resistance R .

A thyristor is a **semiconductor device** with four layers of alternating **P-** and **N-type** materials. It acts exclusively as a **bi-stable** switch, conducting when the gate receives a current trigger, and continuing to conduct until the voltage across the device is reversed biased, or until the voltage is removed (by some other means). It behaves like a controlled diode. Thyristors are widely used in engero-electronic circuits, where they perform the role of switches.

TRIAC (short name for TRIode for Alternating Current, is a three terminal **electronic component** that conducts **current** in either direction when triggered. It acts like thyristor, but the current can flow in both directions. The characteristics of both thyristor and triac are unambiguous with respect to the current of element and depend also on the current of the gate.

6.3. Nonlinear inductors

An inductor, also called a coil or choke is a **passive two-terminal electrical component** that stores energy in a **magnetic field** when **electric current** flows through it. An inductor typically consists of an insulated wire wound into a **coil** around a ferromagnetic core. The general notations of nonlinear inductor, which are used in engineering are shown in Fig. 6.5a [3,16].

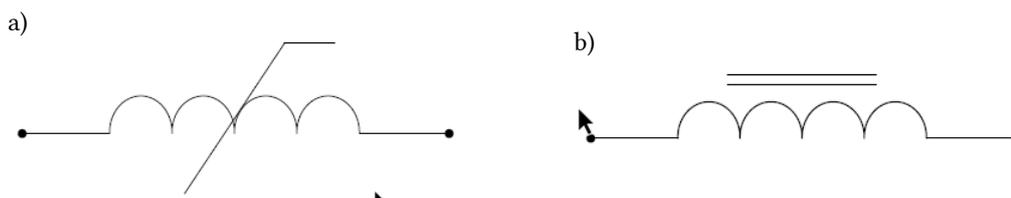


Fig. 6.5 Symbols of nonlinear inductor: a) general symbol, b) symbol of coil built using the ferromagnetic core.

When the current flowing through an inductor changes, the time-varying magnetic field induces an **electromotive force (voltage)** in the conductor, described by Faraday’s law. According to **Lenz’s law**, the induced voltage has a polarity (direction) which opposes the change in current that created it. As a result, inductors oppose any changes in current through them.

The magnetic flux (often denoted as Φ) through a surface is the surface **integral** of the normal component of the **magnetic flux** density B passing through that surface. In the case of many turns of windings we define the flux linkage ψ , which is the total magnetic field passing through the surface (i.e., normal to that surface) formed by a closed conducting loop coil and is determined by the number N of turns in the coil and the magnetic flux, i.e., $\psi=N\Phi$.

The description of inductor associates the flux linkage ψ and current i . If we consider the inductor built on the basis of ferromagnetic core and neglect the hysteresis phenomena the characteristic (ψ, i) may be regarded as monotonic, i.e., both descriptions exist: $\psi=f(i)$ and $i=g(\psi)$. In the case of hysteresis consideration only the general implicit form description of coil $f(\psi, i)=0$ exists.

The voltage-current relation of the nonlinear inductor results from the flux description. Assuming stationary inductor and its flux linkage description in the form $\psi=f(i)$ we get

$$u(t) = \frac{d\psi}{dt} = \frac{d\psi}{di} \frac{di}{dt} = L(i) \frac{di}{dt} \tag{6.15}$$

$$L(i) = \frac{d\psi}{di} \text{ is}$$

The inductance of the coil defined as the derivative of flux linkage with respect to current the nonlinear function of this current. Fig. 6.6 depicts this relations for the coil of the $\psi(i)$ characteristic shown as in the upper curve. Assuming linear flux-current characteristic (linear coil) we get $L(i)=\text{const}$.

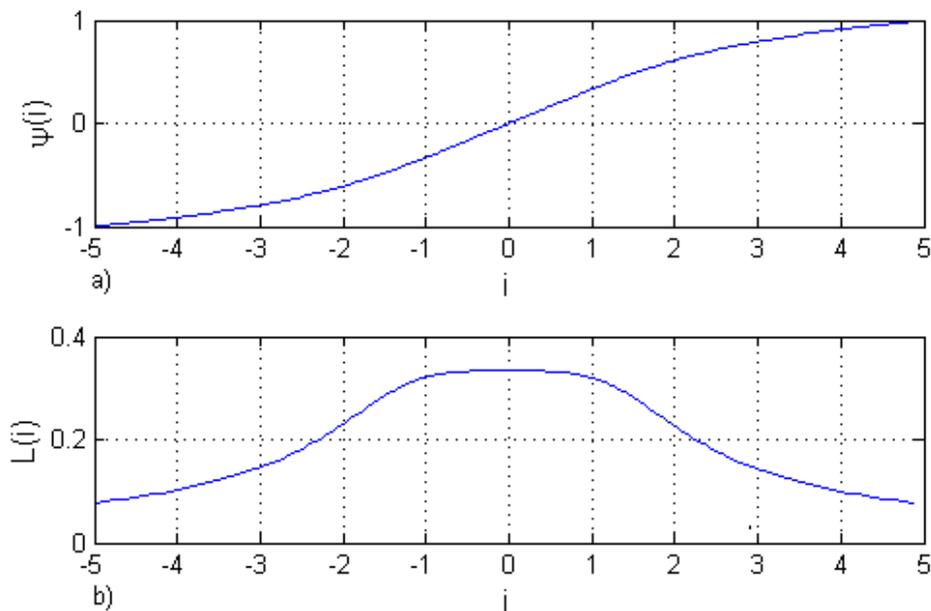


Fig. 6.6. The typical characteristics of the nonlinear coil: upper curve presents the dependence of ψ from i and the bottom curve – the nonlinear inductance $L(i)$.

In the case of complementary description $i=g(\psi)$ we get the following relation

$$\frac{di}{dt} = \frac{di}{d\psi} \frac{d\psi}{dt} = L^{-1}(i) \frac{d\psi}{dt} = L^{-1}(i)u(t) \tag{6.16}$$

The often applied approximation of the coil is of the polynomial form $i=a\psi+b\psi^k$ of the index k depending on the steepness of the characteristic (typical values $k=3, 7, 9$). In such case we have

$$L^{-1}(u) = \frac{di}{d\psi} = a + kb\psi^{k-1} \rightarrow L(i) = \frac{1}{a + kb\psi^{k-1}} \quad (6.17)$$

In the case of non-stationary coil, the flux linkage depends directly on time, i.e., $\psi=f(i,t)$. In such case the voltage-current relations is defined as

$$u(t) = \frac{d\psi(i,t)}{dt} = \frac{\partial f}{\partial i} \frac{di}{dt} + \frac{\partial f}{\partial t} = L(i) \frac{di}{dt} + \frac{\partial f}{\partial t} \quad (6.18)$$

The nonlinear stationary inductor is passive, if the following condition holds:

$$E_L(t) - E_L(t_0) = \int_{t_0}^t u_L(t) i_L(t) dt \geq 0 \quad (6.19)$$

For the inductor controlled by the current this condition may be written in the form

$$E_L(t) - E_L(t_0) = \int_{i(t_0)}^{i(t)} L(i) i_L^2(t) di \geq 0 \quad (6.20)$$

from which it is evident that the inductor is passive if for all values of current the condition $L(i) \geq 0$ holds.

Very often in practice we deal with the multi-terminal inductors. Its general notation including currents and terminal voltages is depicted in Fig. 6.7.

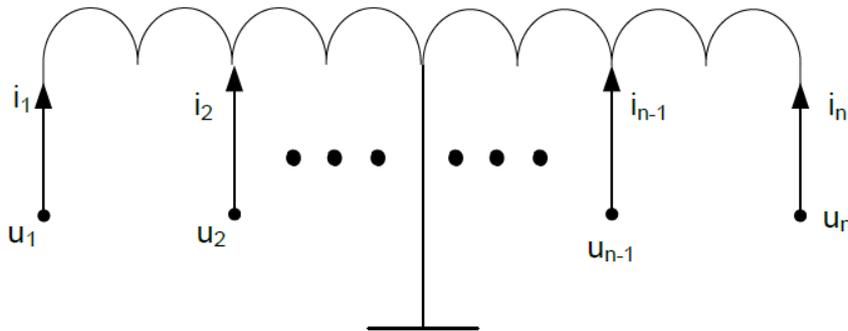


Fig. 6.7 The general notation of the multi-terminal inductor.

If we assume the vector notations of currents, voltages and flux linkages for stationary multi-terminal inductor as shown below

$$\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix}, \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \psi = \begin{bmatrix} \psi_1 = f_1(\mathbf{i}) \\ \psi_2 = f_2(\mathbf{i}) \\ \vdots \\ \psi_n = f_n(\mathbf{i}) \end{bmatrix} \quad (6.26)$$

the voltage-current description of it can be presented as follows

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} df_1/di_1 & df_1/di_2 & \cdots & df_1/di_n \\ df_2/di_1 & df_2/di_2 & \cdots & df_2/di_n \\ \vdots & \vdots & \vdots & \vdots \\ df_n/di_1 & df_n/di_2 & \cdots & df_n/di_n \end{bmatrix} \begin{bmatrix} di_1/dt \\ di_2/dt \\ \vdots \\ di_n/dt \end{bmatrix} = \begin{bmatrix} L_{11}(\mathbf{i}) & L_{12}(\mathbf{i}) & \cdots & L_{1n}(\mathbf{i}) \\ L_{21}(\mathbf{i}) & L_{22}(\mathbf{i}) & \cdots & L_{2n}(\mathbf{i}) \\ \vdots & \vdots & \vdots & \vdots \\ L_{n1}(\mathbf{i}) & L_{n2}(\mathbf{i}) & \cdots & L_{nn}(\mathbf{i}) \end{bmatrix} \begin{bmatrix} di_1/dt \\ di_2/dt \\ \vdots \\ di_n/dt \end{bmatrix}$$

(6.22)

which can be also written in compact matrix form

$$\mathbf{u} = \mathbf{L}(\mathbf{i}) \frac{d\mathbf{i}}{dt} \tag{6.23}$$

$\mathbf{L}(\mathbf{i})$ represents the Jacobian matrix composed of partial dynamic inductances of the coils.

If we need the inductor of the specific type of flux-current characteristic, different from the natural one, corresponding to the ferromagnetic curve, we can apply special circuit built on the basis of mutator loaded by the nonlinear resistor of the needed characteristic, as shown in Fig. 6.8 [4,16]

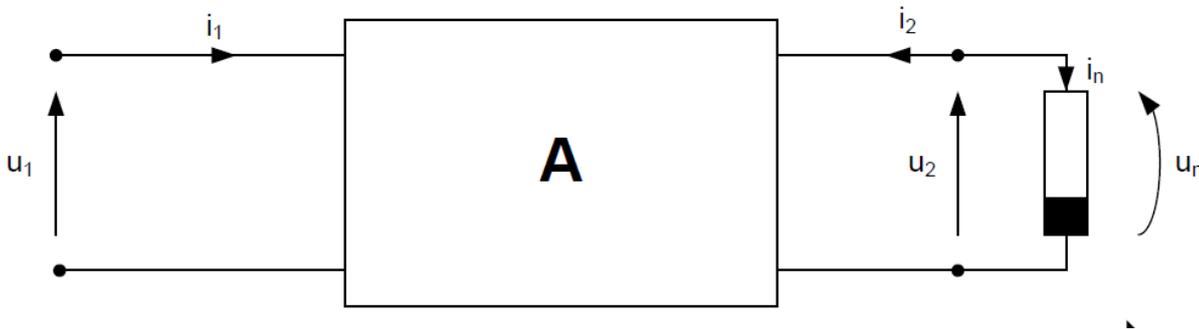


Fig. 6.8 Implementation of nonlinear inductor of the specific characteristic by using the mutator loaded by the nonlinear resistor.

Mutator is a special type of dynamic 2-ports of the very specific chain matrix description

- Converter-type mutator

$$\begin{bmatrix} u_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} k_1 s & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} u_2 \\ -i_2 \end{bmatrix} \tag{6.24}$$

- Inverter-type mutator

$$\begin{bmatrix} u_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} 0 & k_1 s \\ k_2 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ -i_2 \end{bmatrix} \tag{6.25}$$

The coefficients \$k_1\$ and \$k_2\$ represent the scale of transformation and \$s\$ is the differentiation operator. In the case of converter-type mutator we have

$$\begin{aligned} u_1 &= k_1 s u_2 \\ i_1 &= k_2 (-i_2) \end{aligned} \tag{6.26}$$

From the first equation we get the description of the flux linkage \$\psi_1\$

$$\frac{u_1}{s} = k_1 u_2 \rightarrow \int u_1 dt = \psi_1 = k_1 u_2 = k_1 u_N \tag{6.27}$$

Similarly from the second equation (6.26) we have

$$i_1 = k_2 (-i_2) = k_2 i_N \tag{6.28}$$

Hence, from the input terminals of the mutator loaded by the nonlinear resistor $u_N=f(i_N)$ we see the inductor of the following description based on the nonlinear resistor characteristic

$$\psi_1 = k_1 u_N = k_1 f(i_N) = k_1 f\left(\frac{i_1}{k_2}\right) \quad (6.29)$$

The voltage-current relation of the inductor, which is seen from the input terminals of the network is described now in the following form

$$u_L = \frac{d\psi_1}{di_1} \frac{di_1}{dt} = \left(\frac{k_1}{k_2} \frac{df}{di_N} \right) \frac{di_1}{dt} \quad (6.30)$$

It is evident, that the dynamic inductance is described by

$$L(i) = \frac{k_1}{k_2} \frac{df}{di_N} \quad (6.31)$$

Its value depends on the coefficients k_1 and k_2 of mutator and the derivative of the characteristic of the nonlinear resistor with respect to its current.

In the case of the second type of mutator we have the following relations following from the matrix description (6.24)

$$\begin{aligned} u_1 &= k_1 s(-i_2) \\ i_1 &= k_2(u_2) \end{aligned} \quad (6.32)$$

Applying the same mathematical manipulations as for the first type of mutator we get

$$\frac{u_1}{s} = k_1(-i_2) = k_1(i_N) \rightarrow \psi_1 = k_1 i_N \quad (6.33)$$

$$i_1 = k_2 u_2 = k_2 u_N \quad (6.34)$$

Mutator loaded by the nonlinear resistor of the characteristic $i_N=g(u_N)$ implements the inductor described by

$$\psi_1 = k_1 i_N = k_1 g(u_N) = k_1 g\left(\frac{i_1}{k_2}\right) \quad (6.35)$$

The voltage-current relation in the input terminals is given now in the form

$$u_L = \frac{d\psi_1}{di_1} \frac{di_1}{dt} = \left(\frac{k_1}{k_2} \frac{dg}{du_N} \right) \frac{di_1}{dt} \quad (6.36)$$

From which it is evident that the dynamic inductance is described by

$$L(i) = \frac{k_1}{k_2} \frac{dg}{du_N} \quad (6.37)$$

The choice of mutator depends on the type of nonlinear resistor applied in circuit structure, whether it is resistor controlled by the voltage or by the current.

6.3. Nonlinear capacitors

A capacitor is a [passive two-terminal electrical component](#) that stores [potential energy](#) in an [electric field](#). The effect of a capacitor is known as [capacitance](#) C . While some capacitance exists between any two electrical conductors in proximity in a [circuit](#), a capacitor is a special component designed to add capacitance to a circuit. Practical capacitors are available commercially in many different forms. The type of internal dielectric, the structure of the plates and the device packaging all strongly affect the characteristics of the capacitor, and its applications. Values available range from very low (pico-farad range) up to very high values kilo-farad range) in [supercapacitors](#). The general symbol of the nonlinear capacitor is presented in Fig. 6.9.

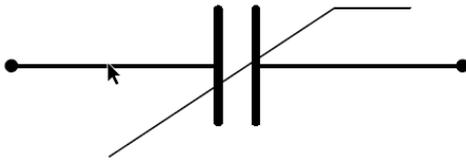


Fig. 6.9 The general symbol of nonlinear capacitor.

The general characteristic of capacitor includes the algebraic relation between the charge $q(t)$ gathered on its plates and the voltage u between the terminals. [6,21]. The typical form of this characteristic represents voltage controlled type, which might be presented as $q=f(u)$ for stationary capacitor and $q=f(u,t)$ for non-stationary one. In the case of monotonic characteristic we can also obtain its inverse in the form $u=f^{-1}(q)$.

The current-voltage relation for stationary capacitor, described by $q=f(u)$, follows from definition of current

$$i(t) = \frac{dq}{dt} = \frac{dq}{du} \frac{du}{dt} = C(u) \frac{du}{dt} \tag{6.38}$$

The dynamic capacitance $C(u) = \frac{dq}{du}$ for nonlinear capacitor is the function of voltage. For example the junction capacitance of the p-n junction in a diode can be expressed as a function of diode voltage u_d

$$C(u_d) = \frac{C_{j0}}{\left(1 - \frac{u_d}{u_j}\right)^m} \tag{6.39}$$

The parameter C_{j0} is the capacity of junction at no polarization, u_j – threshold voltage, typically in the range $(0,6 - 1)V$, m – exponent of typical value in the range $(0.3 - 0.5)$ depending on the type of junction.

In the case of nonstationary capacitor described by $q=f(u,t)$ we have

$$i(t) = \frac{dq(u,t)}{dt} = \frac{\partial f}{\partial u} \frac{du}{dt} + \frac{\partial f}{\partial t} = C(u) \frac{du}{dt} + \frac{\partial f}{\partial t} \tag{6.40}$$

As an example, if the charge-voltage of the nonstationary capacitor is expressed in the form

$$q = A(\sin \omega t)^2 \cdot u^3$$

the current of capacitor is given by

$$i(t) = \frac{dq(u,t)}{dt} = \frac{\partial f}{\partial u} \frac{du}{dt} + \frac{\partial f}{\partial t} = \left[3A(\sin \omega t)^2 u^2\right] \frac{du}{dt} + A\omega \sin(2\omega t) u^3$$

From the general condition of passivity for capacitor it follows

$$E_C(t) - E_C(t_0) = \int_{t_0}^t u_C(t) i_C(t) dt \geq 0 \tag{6.41}$$

Assuming stationarity, this condition can be written in the form

$$E_C(t) - E_C(t_0) = \int_{u(t_0)}^{u(t)} C(u) u_C^2(t) du \geq 0 \tag{6.42}$$

If $C(u) = \frac{dq}{du}$ is positive in the whole range of voltage, the capacitor is passive.

Often we deal with a multi-terminal capacitors. Denoting currents, voltages and charges at the terminals in a vector form

$$\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix}, \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \mathbf{q} = \begin{bmatrix} q_1 = f_1(\mathbf{u}) \\ q_2 = f_2(\mathbf{u}) \\ \vdots \\ q_n = f_n(\mathbf{u}) \end{bmatrix} \quad (6.43)$$

The current-voltage relations for multi-terminal capacitor may be written as follows

$$\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix} = \begin{bmatrix} df_1/du_1 & df_1/du_2 & \cdots & df_1/du_n \\ df_2/du_1 & df_2/du_2 & \cdots & df_2/du_n \\ \vdots & \vdots & \vdots & \vdots \\ df_n/du_1 & df_n/du_2 & \cdots & df_n/du_n \end{bmatrix} \begin{bmatrix} du_1/dt \\ du_2/dt \\ \vdots \\ du_n/dt \end{bmatrix} = \begin{bmatrix} C_{11}(\mathbf{u}) & C_{12}(\mathbf{u}) & \cdots & C_{1n}(\mathbf{u}) \\ C_{21}(\mathbf{u}) & C_{22}(\mathbf{u}) & \cdots & C_{2n}(\mathbf{u}) \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1}(\mathbf{u}) & C_{n2}(\mathbf{u}) & \cdots & C_{nn}(\mathbf{u}) \end{bmatrix} \begin{bmatrix} du_1/dt \\ du_2/dt \\ \vdots \\ du_n/dt \end{bmatrix} \quad (6.44)$$

or in the compact matrix form

$$\mathbf{i} = \mathbf{C}(\mathbf{u}) \frac{d\mathbf{u}}{dt} \quad (6.45)$$

where $\mathbf{C}(\mathbf{u})$ is a Jacobian matrix describing the partial dynamic capacitances with respect to elements of vector \mathbf{u} [3,16].

In the case of realization of the nonlinear capacitor of the strictly specified nonlinear characteristic of resistor we can use the same structure as for inductor, presented in Fig. 6.8 and composed of mutator loaded by the nonlinear resistor. Two types of mutators can be used here:

- Converter-type mutator

$$\begin{bmatrix} u_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 s \end{bmatrix} \begin{bmatrix} u_2 \\ -i_2 \end{bmatrix} \quad (6.46)$$

- Inverter-type mutator

$$\begin{bmatrix} u_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} 0 & k_1 \\ k_2 s & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ -i_2 \end{bmatrix} \quad (6.47)$$

From the chain description of (6.46) we get in the converter-type mutator

$$\begin{aligned} u_1 &= k_1 u_2 = k_1 u_N \\ i_1 &= k_2 s(-i_2) = k_2 s i_N \end{aligned} \quad (6.48)$$

From these equations for the resistor controlled by the voltage $i_N = g(u_N)$ we obtain

$$\frac{i_1}{s} = k_2 i_N \rightarrow \int i_1 dt = q_1 = k_2 i_N \quad (6.49)$$

$$q_1 = k_2 i_N = k_2 g(u_N) = k_2 g\left(\frac{u_1}{k_1}\right) \quad (6.50)$$

As it is seen the nonlinear characteristic of capacitor depends now from the nonlinear resistor and parameters k_1 and k_2 of mutator. The current voltage relation in the input terminals is described by

$$i_C = \frac{dq_1}{du_1} \frac{du_1}{dt} = \left(\frac{k_2}{k_1} \frac{dg}{du_N} \right) \frac{du_1}{dt} \quad (6.51)$$

The nonlinear capacitance $C(u)$ is the equal

$$C(u) = \frac{k_2}{k_1} \frac{dg}{du_N} \quad (6.52)$$

Similarly for the second type of mutator we have

$$\begin{aligned} u_1 &= k_1(-i_2) = k_1 i_N \\ i_1 &= k_2 s u_2 = k_2 s u_N \end{aligned} \quad (6.53)$$

Applying the same mathematical manipulations for nonlinear resistor controlled by the current $u_N=f(i_N)$ we obtain

$$\frac{i_1}{s} = k_2 u_N \rightarrow q_1 = k_2 u_N \quad (6.54)$$

$$q_1 = k_2 u_N = k_2 f(i_N) = k_2 f\left(\frac{u_1}{k_1}\right) \quad (6.55)$$

The current voltage relation in the input terminals is described by

$$i_C = \frac{dq_1}{du_1} \frac{du_1}{dt} = \left(\frac{k_2}{k_1} \frac{df}{di_N} \right) \frac{du_1}{dt} \quad (6.56)$$

from which the relation describing the implemented capacitance follows

$$C(i) = \frac{k_2}{k_1} \frac{df}{di_N} \quad (6.57)$$

The capacitor of any type of nonlinearity can be implemented by using the nonlinear resistor of the specified characteristic and proper type of mutator .

6.4. Memristors

A memristor (called memory resistor) is a **hypothetical** non-linear **passive two-terminal electrical component** relating **electric charge** q and magnetic **flux linkage** ψ . It was envisioned, in 1971 by prof. **Leon Chua** [2]. According to the characterizing mathematical relations, the memristor would hypothetically operate in the following way. The memristor's **electrical resistance** is not constant but depends on the history of current that had previously flowed through the device, i.e., the device remembers its history. When the electric power supply is turned off, the memristor remembers its most recent resistance until it is turned on again.

The memristors are intended for applications in **nanoelectronic** memories, computer logic and **neuromorphic/neuromemristive** computer architectures. In 2013, Hewlett-Packard suggested that memristor memory may become commercially available as early as 2018. In March 2012, a team of researchers from **HRL Laboratories** and the **University of Michigan** announced the first functioning memristor array built on a **CMOS** chip. The graphical symbol of memristor proposed by Chua is presented in Fig. 6.10 [2].

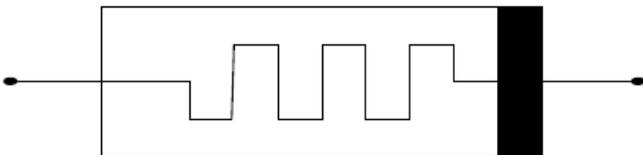


Fig. 6.10. Symbol of nonlinear memristor.

The general description of the stationary memristor may be presented in an implicit form $f(\psi, q)=0$. We can define two types of memristors from this relation.

- memristor controlled by the charge

$$\psi = f(q) \quad (6.58)$$

- memristor controlled by the flux linkage

$$q = g(\psi) \quad (6.59)$$

The current-voltage relations for memristor represents the form typical for resistors

$$u = \frac{d\psi}{dt} = \frac{d\psi}{dq} \frac{dq}{dt} = M(q)i \quad (6.60)$$

The variable $M = \frac{d\psi}{dq}$ means memristance, which is equivalent to the charge dependent resistance of the

$$W = \frac{dq}{d\psi}$$

resistor. The inverse of memristance is called memductance and defined as $W = \frac{dq}{d\psi}$. For linear memristors we have $M=1/W$. The relation $u(t) = M(q)i(t)$ implies time varying charge. Furthermore, the memristor is static if no current is applied. If $i(t) = 0$, we find $u(t) = 0$ and $M(t)$ is constant. This is the essence of the memory effect. The [power consumption](#) description is typical for resistor, i.e.,

$$p(t) = u(t)i(t) = M(q)i^2(t) \quad (6.61)$$

As long as $M(q(t))$ varies little, such as under alternating current, the memristor will appear as a constant resistor. $M(q)$ is physically restricted to be positive for all values of q (assuming the device is passive and does not become [superconductive](#) at some q).

In 2008 researchers from [HP Labs](#) [5] introduced a model for a memristance function based on thin films of [titanium dioxide](#). For $R_{ON} \ll R_{OFF}$ the memristance function was determined as follows

$$M(q) = R_{off} \left(1 - \frac{\mu_v R_{on}}{D^2} q(t) \right)$$

where R_{off} represents the high resistance state, R_{on} – the low resistance state, μ_v – the mobility of dopants in the thin film, and D – the film thickness.

Example 6.2

To illustrate the performance of memristor consider the circuit structure presented in Fig. 6.11a. Let us determine the current $i(t)$ and voltage $u_0(t)$ in this circuit assuming $R_1=2\Omega$, $R_2=1\Omega$. The piece-wise linear characteristic of the hypothetical memristor is shown in Fig. 6.11b and the supply voltage $e(t)$ has the form of Fig. 6.11c.

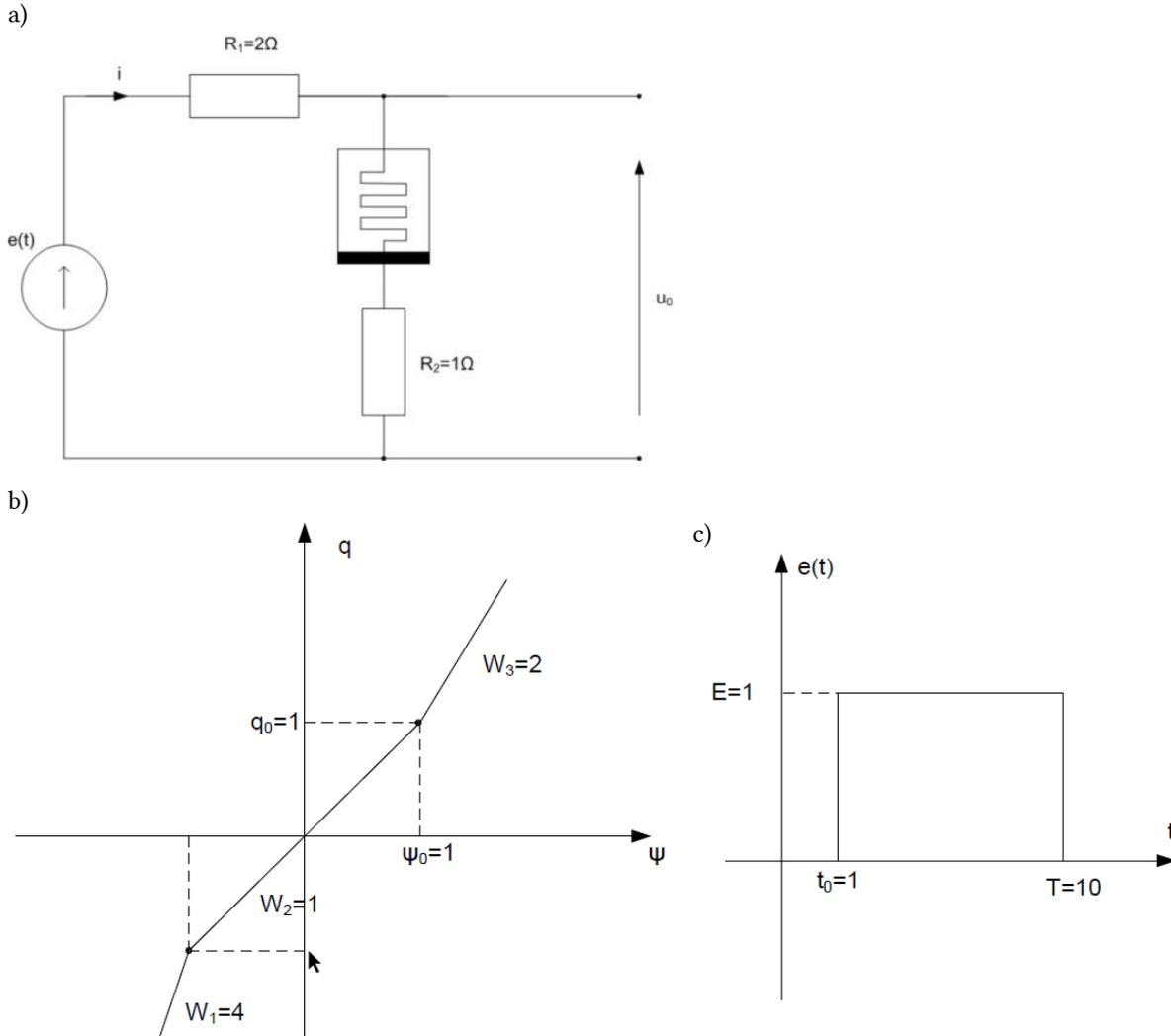


Fig. 6.11 The structure of the memristive circuit (a), charge-flux characteristic of memristor (b) and the form of the circuit excitation (c).

The dynamic memristance in the particular regions of the flux linkage is equal

$$M(q) = \frac{1}{W(q)} = \begin{cases} 0,25 & q < -1 \\ 1 & -1 < q < 1 \\ 0,5 & q > 1 \end{cases}$$

At positive values of excitation the memristor can operate only in the positive ranges of charge, which correspond to memductance W_2 or W_3 (memristance M_2 or M_3). The current in the circuit behaves like for purely resistive elements, i.e.,

$$i = \frac{dq}{dt} = \frac{E}{R_1 + R_2 + M(q)}$$

and the output voltage u_0 is described by the relation

$$u_0 = e - R_1 i$$

In the particular region corresponding to the memristance M_2 we have

$$i = I_1 = \frac{E}{R_1 + R_2 + M_2}$$

$$u_0 = E \frac{R_2 + M_2}{R_1 + R_2 + M_2}$$

Similarly in the case of operation in M_3 region we get

$$i = I_2 = \frac{E}{R_1 + R_2 + M_3}$$

$$u_0 = E \frac{R_2 + M_3}{R_1 + R_2 + M_3}$$

Changing the operation point from M_2 to M_3 will happen in the time point T_0 , when the electric charge of memristor achieves the value q_0 . From the relations between current and charge we get

$$q = \int_0^{T_0} i dt = \int_0^{T_0} I_1 dt = I_1 T_0 = q_0 \rightarrow T_0 = \frac{q_0}{I_1} = \frac{q_0(R_1 + R_2 + M_2)}{E}$$

At the given values of elements we get

$$I_1 = \frac{E}{R_1 + R_2 + M_2} = 0,25 \rightarrow u_0 = 0,75$$

$$I_2 = \frac{E}{R_1 + R_2 + M_3} = 0,286 \rightarrow u_0 = 0,375$$

$$T_0 = 4$$

Fig. 6.12 shows the current $i(t)$ in the circuit and the form of output voltage $u_0(t)$ as a function of time.

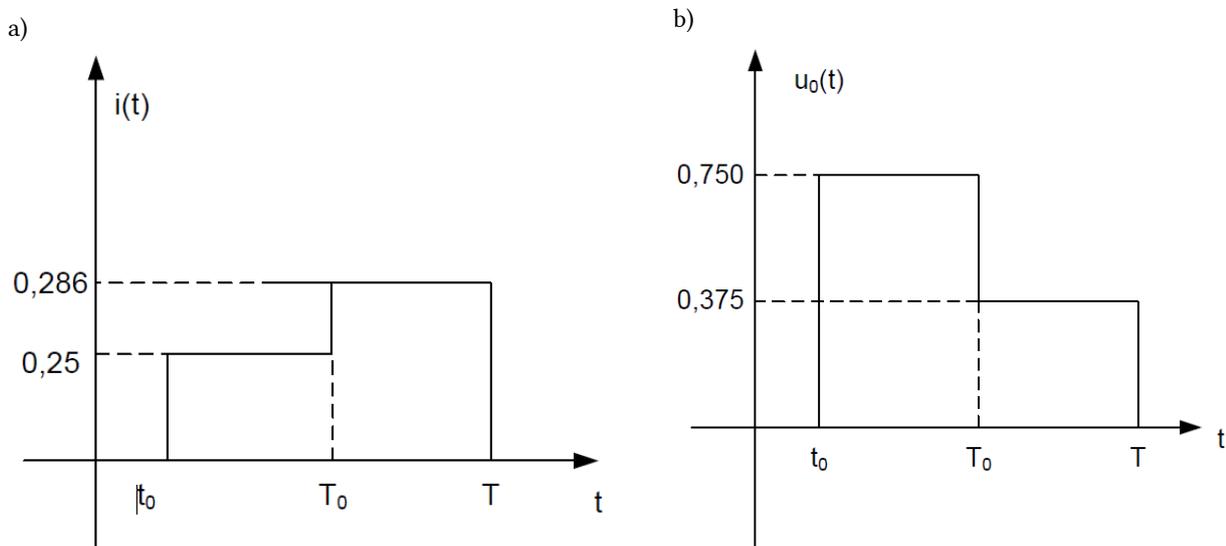


Fig. 6.12. The time form of: a) current $i(t)$ in the circuit and b) output voltage $u_0(t)$.

6.5 Exercises

1. Calculate and plot the changes of dynamic parameter of the nonlinear element of the given characteristic $f(x)$ shown in Fig. 6.13a.

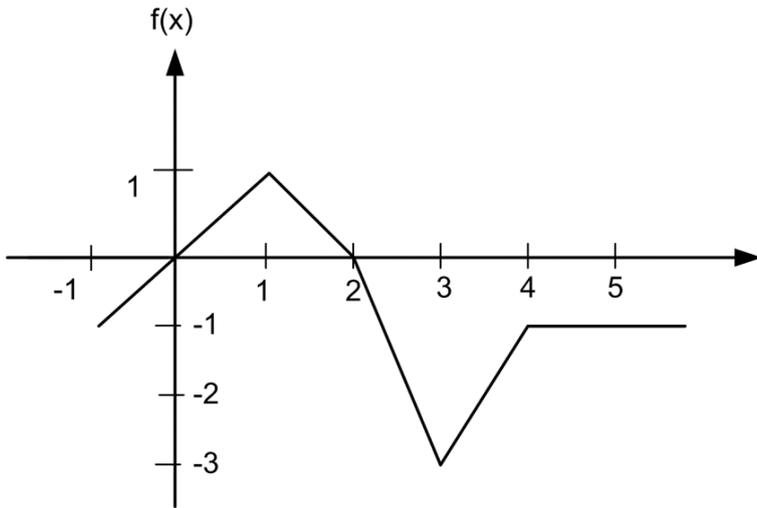


Fig. 6.13a The characteristic of nonlinear element

Solution:

The dynamic parameter D in the operating point x_0 is described by the equation

$$D(x_0) = \left. \frac{dy}{dx} \right|_{x=x_0} = \left. \frac{df(x)}{dx} \right|_{x=x_0}$$

These values are calculated for all piecewise linear regions and the dynamic parameter changes in all subregions are shown in Fig. 6.13b.

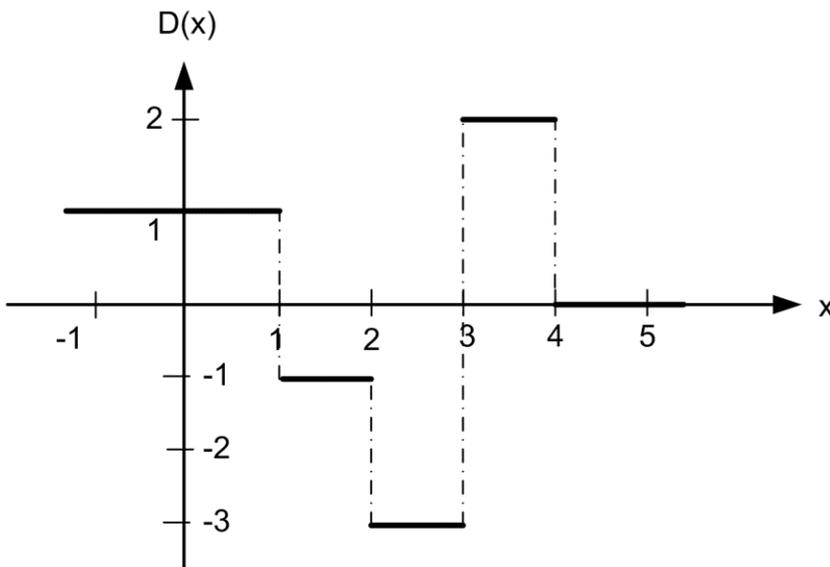


Fig. 6.13b The plot of dynamic parameter of the nonlinear element.

1. The nonlinear inductor is described by

$$i = a\psi + b\psi^3$$

where typical values of the parameters are: $a=10.23A/Wb$, $b=2.4A/Wb^3$. Calculate and plot the dynamic inductance of the inductor.

Solution:

Let us differentiate current with respect to time. Hence we get

$$\frac{di}{dt} = \frac{di}{d\psi} \frac{d\psi}{dt} = (a + 3b\psi^2) \frac{d\psi}{dt}$$

From this we get the expression for inductance L

$$L = \frac{d\psi}{di} = \frac{1}{(a + 3b\psi^2)}$$

Fig. 6.2 presents the characteristics of flux and inductance as a function of current.

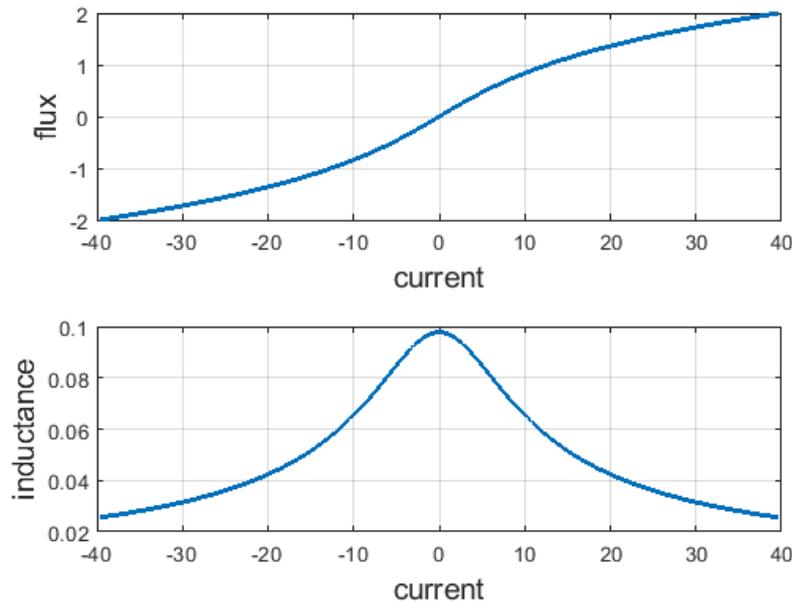


Fig. 6.14 The nonlinear characteristics of the inductor

1. Junction diode current is described by the equation

$$i_d = I_0 \left(e^{u_d/V_T} - 1 \right)$$

where u_d , i_d represent the voltage and current of the diode, respectively, $I_0=10^{-6}\text{A}$ and $V_T=26\text{mV}$. Determine the dynamic admittance Y of the diode and plot it for $0 < u_d < 0.7\text{V}$.

Solution:

After differentiation of the formula for the current we get

$$Y(u_d) = \frac{di_d}{du_d} = \frac{I_0}{V_T} e^{u_d/V_T}$$

The change of admittance with the voltage u_d is plotted in Fig. 6.3.

It is evident that for positive polarity of voltage the admittance value increases rapidly (the impedance decreases). For the supplying voltage above 0.5V the diode can be treated in practice as the short circuit.

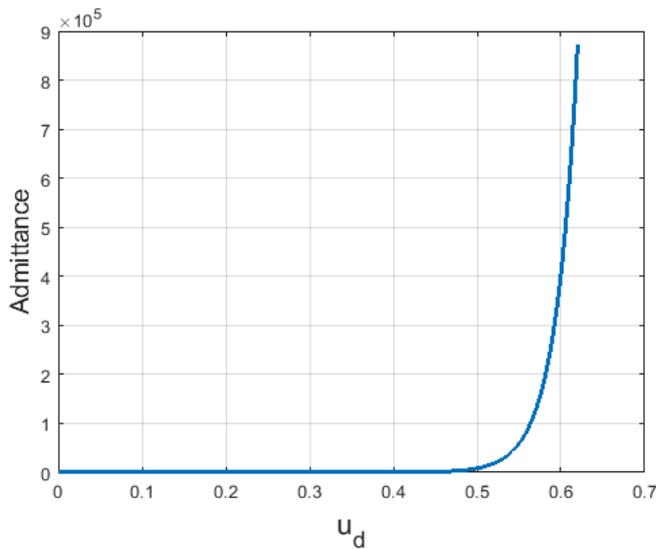


Fig. 6.15 The change of admittance of the diode with the voltage u_d

1. The piecewise linear characteristic of memristor is presented below. Determine the memristance and menductance of this element.

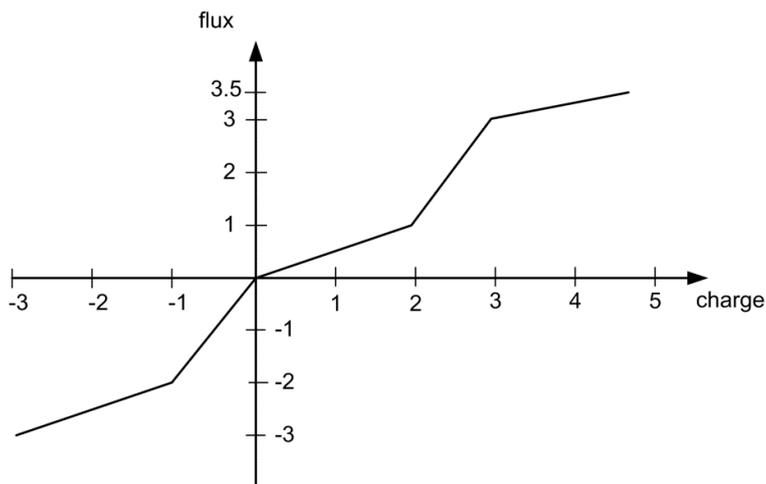


Fig. 6.16 The flux-charge characteristic of the memristor

Solution:

$$M = \frac{d\psi}{dq}$$

The memristance of memristor is defined as $M = \frac{d\psi}{dq}$. Taking into account the piecewise linear character of the memristor characteristic we get

$$M = \begin{cases} 1/2 & \text{for } q < -1 \\ 2 & \text{for } -1 < q < 0 \\ 1/2 & \text{for } 0 < q < 2 \\ 2 & \text{for } 2 < q < 3 \\ 1/4 & \text{for } q > 3 \end{cases}$$

Similarly, the mendentance is defined as $W = \frac{dq}{d\psi}$. Hence, we get

$$W = \begin{cases} 2 & \text{for } q < -1 \\ 1/2 & \text{for } -1 < q < 0 \\ 2 & \text{for } 0 < q < 2 \\ 1/2 & \text{for } 2 < q < 3 \\ 4 & \text{for } q > 3 \end{cases}$$

6.6. Basic definitions

Bipolar transistor – three terminal [semiconductor device](#) composed of three layers of the terminals: base, collector and emitter. Bipolar transistors use two junctions between two [semiconductor](#) types, n-type and p-type and are manufactured as NPN and PNP. A [voltage](#) or [current](#) applied to one pair of the transistor's terminals controls the current through another pair of terminals.

Diac – the diode that conducts electrical current only after its breakover voltage, has been reached momentarily. The diode remains in conduction until the current through it drops below a value characteristic for the device, called the holding current. Below this value, the diode switches back to its high-resistance, non-conducting state.

Diode – the two-terminal [electronic device](#) that conducts [current](#) primarily in one direction (asymmetric [conductance](#)); it has low (ideally zero) [resistance](#) in one direction, and high (ideally infinite) [resistance](#) in the other.

Dynamic parameter – the derivative of the function $f(x)$ describing nonlinear element with respect to its argument x in the actual operating point. It measures the sensitivity to change of the function value (output value) with respect to a change in its argument (input value).

Field effect transistor – [transistor](#) that uses an [electric field](#) to control the electrical behavior of the device. They involve single-carrier-type operation. There exist many different implementations of field effect transistors (JFET, MOSFET, GASFET, etc.). They generally display very [high input impedance](#) at low frequencies. The [conductivity](#) between the drain and source terminals is controlled by an electric field in the device, which is generated by the voltage difference between the source and the gate of the device.

Memristance – the derivative of the flux with respect to the charge of memristor,

$$M = \frac{d\psi}{dq}.$$

Menductance – the derivative of the charge with respect to the flux of memristor,

$$W = \frac{dq}{d\psi}.$$

Mutator – the dynamic 2-port used to transform the nonlinear resistor into nonlinear capacitor or nonlinear inductor.

Nonlinear capacitor – the element for which the charge is dependent in nonlinear way from the voltage on its terminals. Its general description involves the charge-voltage relation (q, u) and voltage-current relation involves the nonlinear differential equation.

Nonlinear element – electrical device which does not have a linear relationship between current and voltage. The parameters of it are dependent on input variable, either current or voltage. Hence, we recognize element controlled by the voltage or current.

Nonlinear inductor – the element for which the voltage drop across the inductor is not linearly proportional to the rate of change of current, such as when the inductor has a core of magnetic material in

which magnetic induction is not proportional to magnetic field strength. Its general description involves the flux-current relation (ψ, i) and voltage-current relation involves the nonlinear differential equation.

Nonlinear resistor – the nonlinear element, which is fully defined by the voltage-current relationship expressed as the nonlinear algebraic function. No differential or difference equation is involved in its description.

Nonlinear memristor – the hypothetical non-linear passive two-terminal electrical component relating electric charge q and magnetic flux linkage ψ . Its general description requires the nonlinear relation charge-flux (ψ, q).

Static parameter – the ratio of the value of the function $f(x)$ describing nonlinear element to the value of the independent variable x .

Thyristor – the semiconductor device with four layers of alternating P- and N-type materials. It acts exclusively as a bi-stable switch, conducting when the gate receives a current trigger, and continuing to conduct until the voltage across the device is reversed biased, or until the voltage is removed (by some other means). It behaves like a controlled diode.

Triac – the three terminal electronic element that conducts current in either direction when triggered. It acts like thyristor, but the current can flow in both directions.

Tunnel diode – the type of semiconductor diode that has negative resistance due to the quantum mechanical effect called tunneling. In certain region of characteristic the electric current decreases as the voltage increases.

Varistor – the electronic device with the resistance that varies with the applied voltage. Its current-voltage is similar to that of a diode. However, in contrast to a diode, it has the same characteristic for both directions of traversing current. At low voltage it has a high electrical resistance which decreases as the voltage is raised.

Zener diode – the particular type of diode that allows current to flow not only from its anode to its cathode, but also in the reverse direction, when the Zener voltage is reached. By contrast with the conventional diode, a reverse-biased Zener diode exhibits a controlled breakdown and allows the current to keep the voltage across the Zener diode close to the Zener breakdown voltage (stabilization of voltage).

7. Analysis of nonlinear resistive circuits

7.1. Introduction

The main difference, which is very important in analysis of nonlinear circuits, is impossibility of applying the superposition theorem [3]. It means, that the analysis should be performed at the simultaneous existence of all sources of different character. It complicates the computation process. At the same time the circuit may show many different solutions, depending on the starting point of analysis and in general we never know what is their number. Additional problem is lack of general analytical methods of solution of the set of nonlinear equations, which is in a great contrast to the linear systems, for which many algebraic methods exist.

This chapter will be devoted to selected methods of analysis of nonlinear resistive circuits only at DC excitation. The sinusoidal sources should be analyzed step by step for succeeding points of time, distant from each other by the assumed increment.

The formulation of the description of the complex nonlinear circuit is performed by applying the Kirchhoff's laws and description of the characteristics of nonlinear elements. As a result of such approach we get the system of nonlinear equations, which can be written in the general form $\mathbf{f}(\mathbf{u}, \mathbf{i}) = \mathbf{0}$, where \mathbf{u} and \mathbf{i} are the vectors of voltages and currents of circuit elements, respectively.

This set of equations is then subject to solution. Unfortunately, in general case there are no analytical, algebraic methods of solution of the system of nonlinear equations. Only in very special cases they exist. To such methods belong the graphical methods and graph-analytical methods applying the piece-wise linearization of nonlinear characteristics. The only general method, which can deliver the solution of the system of nonlinear equations are the iterative numerical methods, based on Newton algorithm. This chapter will present some chosen approaches to the solution of nonlinear circuits.

7.2. Selected methods of solution of nonlinear circuits

7.2.1. Graphical methods

The graphical methods operate directly on the characteristics of the elements. The analysis aims on simplification of the circuit by finding equivalent characteristics of series and parallel connected elements [16].

The current flowing through the elements connected in series is the same, while total equivalent voltage is the sum of the voltages of elements. It means that these operations may be written in the form

$$\begin{aligned}i &= i_1 = i_2 \\u &= u_1 + u_2\end{aligned}\tag{7.1}$$

where the variables (u, i) mean, respectively, voltage and current of the equivalent element and (u_1, i_1) , (u_2, i_2) the corresponding variables of the first and second element. The addition of voltages is done for the current

points common to both nonlinear elements. Having the equivalent characteristic of the circuit supplied by the voltage or current source we can read from this characteristic the lacking variable (current in the case of voltage excitation and voltage at current source).

Example 7.1

The example of two nonlinear elements connected in series and supplied by the voltage source is shown in Fig. 7.1a. The current-voltage characteristics and their equivalent of the series connection are presented in Fig. 7.1b. The characteristic denoted by u_1+u_2 was obtained by adding voltages in vertical direction for the succeeding values of current.

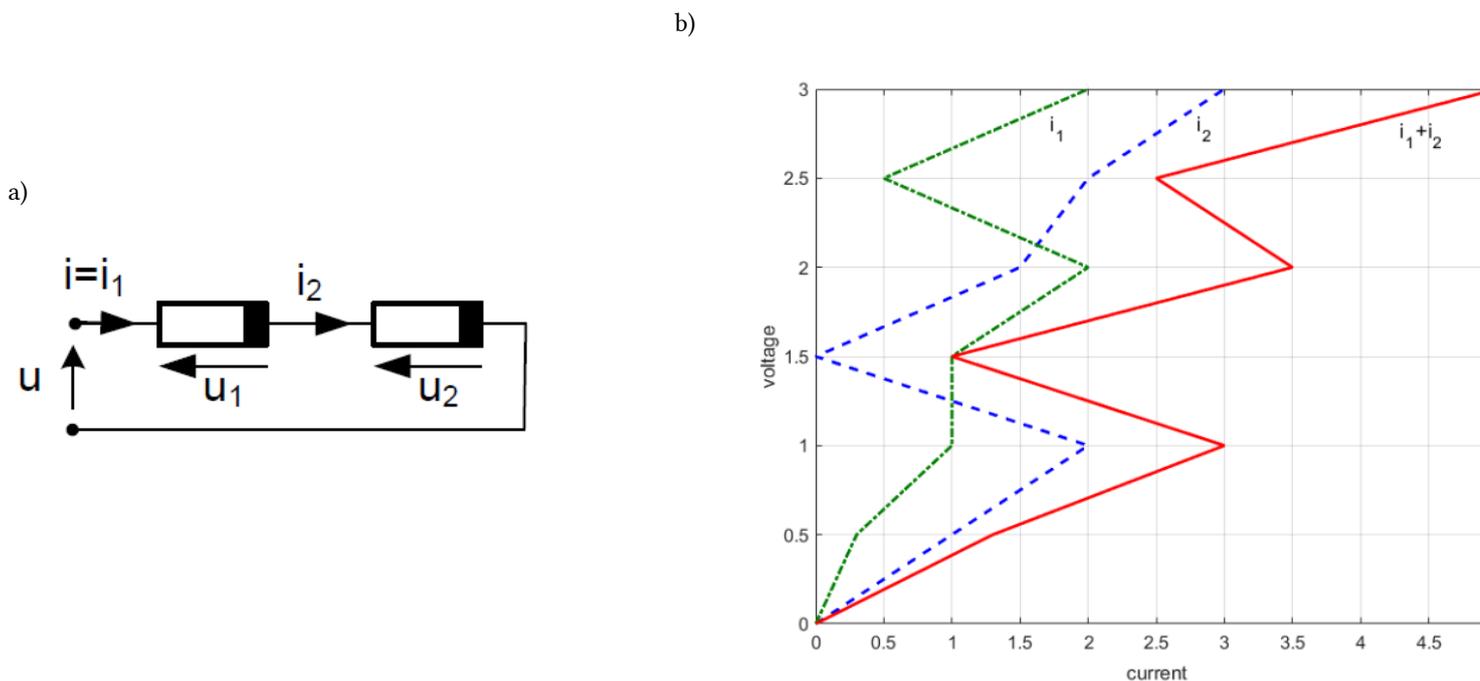


Fig. 7.1. Example of two nonlinear elements connected in series: a) the circuit configuration, b) the characteristics of elements and their equivalent characteristic.

From the equivalent characteristic (red) at given value of supply voltage u we can read the current i , which flows in both resistors. Having this value we can immediately read the voltage of both resistors from their respective characteristics.

In the case of parallel connection of nonlinear elements the voltage of both elements is common, while the equivalent current is the sum of currents of the individual resistors. It means, that the equations describing such connection may be written as follows

$$\begin{aligned}
 u &= u_1 = u_2 \\
 i &= i_1 + i_2
 \end{aligned}
 \tag{7.2}$$

where the variables (u, i) are the voltage and current of the equivalent element. The addition of currents should be done point by point at constant value of the voltage u of both nonlinear elements.

Example 7.2

The example of parallel connection of two nonlinear elements is shown in Fig. 7.2a. Their current-voltage

characteristics as well as their equivalent are presented in Fig. 7.2b. The characteristic denoted by i_1+i_2 was obtained by adding currents in horizontal direction for the succeeding values of voltage.

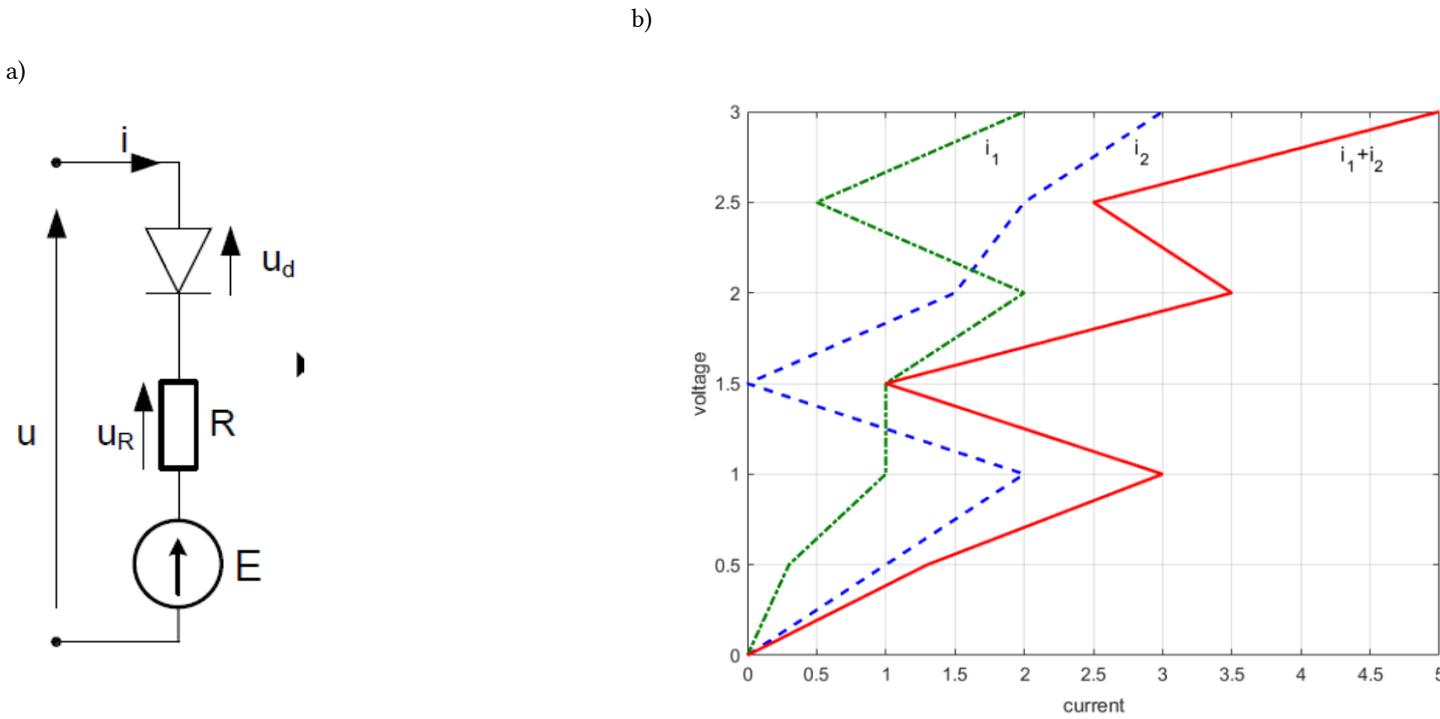


Fig. 7.2. . Example of two nonlinear elements connected in parallel: a) the circuit configuration, b) the characteristics of elements and their equivalent characteristic.

Example 7.3

Another practical example of application of the graphical methods will be presented for diode-resistor circuit, connected in series with the ideal voltage source shown in Fig. 7.3a. The ideal diode characteristic is treated as either short circuit (when the diode voltage is non-negative) or open circuit (when the diode voltage is negative).

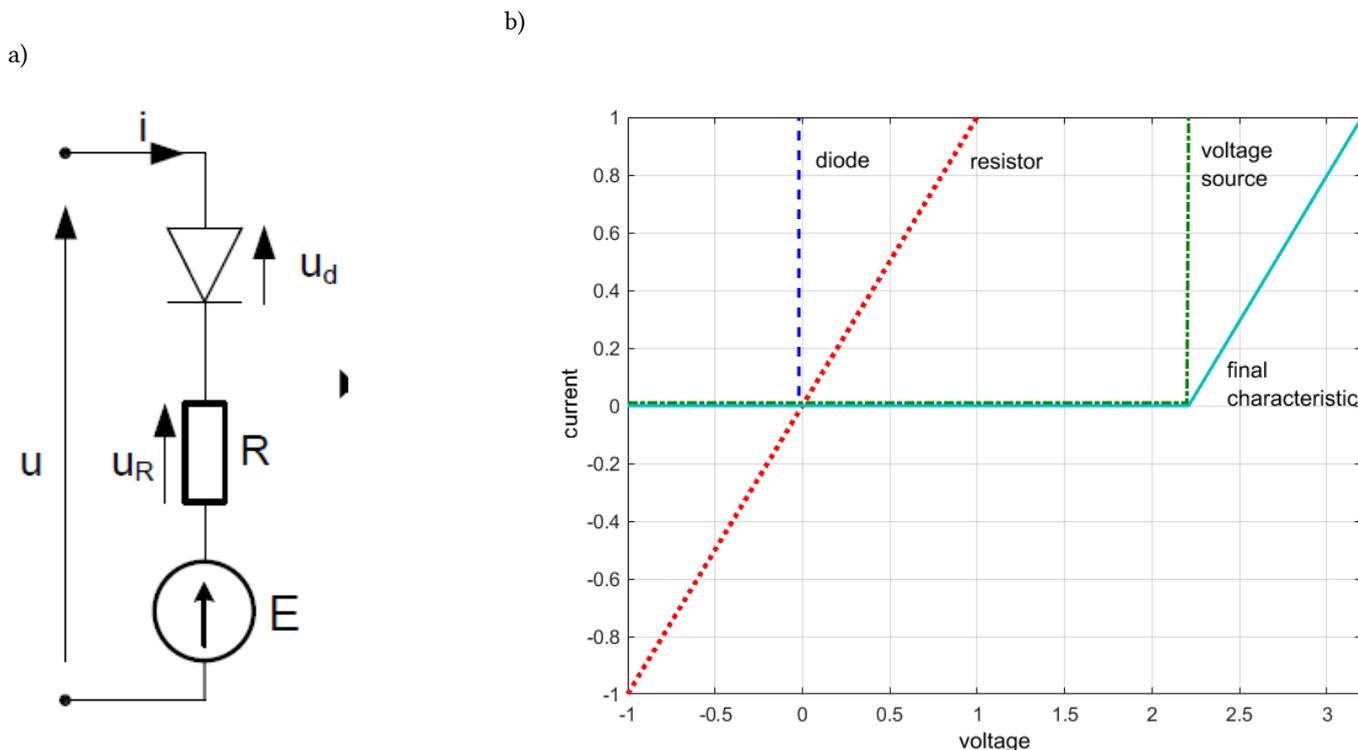


Fig. 7.3. Example of diode circuit configuration (a) and construction of the equivalent characteristic (b).

As a result of adding these three characteristics (diode + resistor + ideal voltage source) we get their equivalent, representing the circuit from the input terminals point of view (the continuous line).

7.2.2. Graphical-analytical methods based on piece-wise linearization

This method applies piece-wise linear (PWL) approximation of the nonlinear characteristics. Each linear section of characteristic is represented by the linear elements (resistor and source).

Generally speaking, piecewise-linear models have universal approximation properties, which means that any nonlinear function can be approximated by a PWL function with arbitrary accuracy, provided that the function domain is partitioned in a large enough number of subdomains where the function is approximated in a linear way [3,16].

In this way the nonlinearity disappears and the circuit is fully piece-wise linear, easy for analysis using any known method from linear circuit theory. However, many such PWL circuits may be formed. As a result of analysis of these circuits we obtain many solutions determining voltages (vector $\mathbf{u}=[u_1, u_2, \dots, u_n]T$) and currents (vector $\mathbf{i}=[i_1, i_2, \dots, i_n]T$) of the particular branches of the circuit. Not all of them can be accepted, since some solutions may be not located in the considered piece-wise sections of the nonlinear resistor. The only accepted solutions are related to the balance (equilibrium) points, in which the values of current and voltage found in solution, agree with the actually considered section of the PWL approximation of nonlinear element.

This approach is very easy for the circuit containing only one nonlinear element. In such case, according to Thevenin theorem, the complex linear part of the circuit can be represented from the terminals of nonlinear element by the series connection of the equivalent resistance R_{AB} and ideal voltage source U_{AB} . The nonlinear element described by $u_N=f(i_N)$ is then added to this Thevenin equivalent circuit forming in this way the one-loop circuit, which is easy in analysis. Such situation is illustrated in Fig. 7.4.

Further analysis needs drawing the characteristics for linear and nonlinear parts of the circuit in common plot.

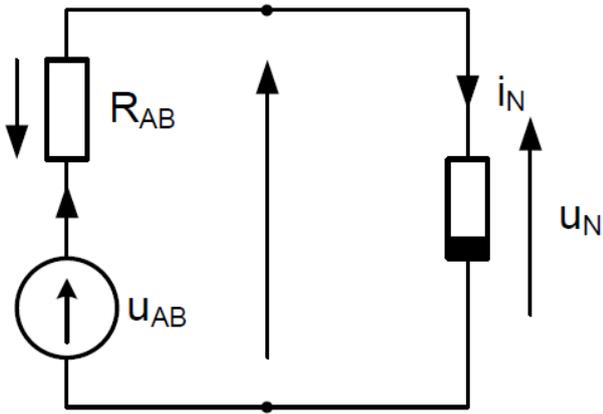


Fig. 7.4. One-loop circuit with the nonlinear element.

The crossing points of the load line of the linear part and the nonlinear characteristic show the possible sections of the nonlinear characteristic, which should be taken into account in the analysis.

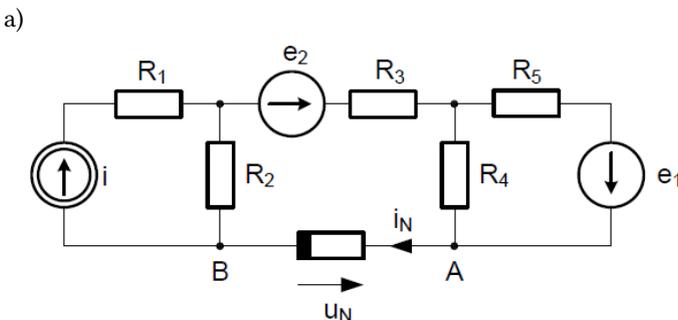
Observe, that in one loop circuit there is only one (common) current i_N of linear and nonlinear part of the circuit. Moreover, the voltage u_N is also appearing in a linear part of the circuit (because of parallel connection). Therefore, the equilibrium point can be determined from the Kirchhoff's law, which is written in the form

$$u_{AB} - R_{AB}i_N = u_N = f(i_N) \tag{7.3}$$

The solution of this equations exists in the crossing point of the linear part of circuit described by $u_{AB} - R_{AB}i_N = u_N$ and the original nonlinear characteristic $u_N = f(i_N)$. Applying PWL representation of this characteristic we transform the equation (7.3) to the linear equation of unique solution in each section.

Example 7.3

As an example consider the nonlinear circuit structure presented in Fig. 7.5a. Assume following values of parameters: $R_1=15\Omega$, $R_2=5\Omega$, $R_3=8\Omega$, $R_4=4\Omega$, $R_5=6\Omega$, $e_1=50V$, $e_2=20V$, $i=10A$. Fig. 7.5b represents the characteristic of the nonlinear element.



b)

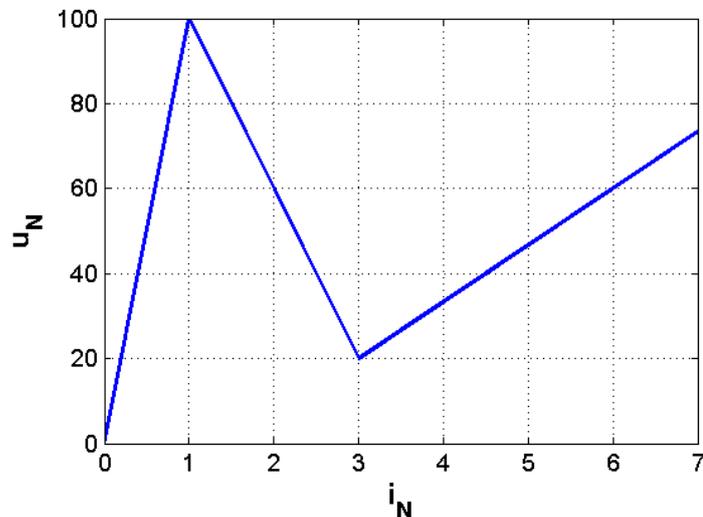


Fig. 7.5. Example of analysis of nonlinear resistive circuit: a) circuit structure, b) characteristic of nonlinear element.

The equivalent Thevenin circuit representing the linear part from the A-B terminals, is shown in Fig. 7.6a. The values of the elements are as follows:

$$R_{AB} = R_2 + R_2 + \frac{R_4 R_5}{R_4 + R_5} = 15,4\Omega$$

$$u_{AB} = e_2 + R_2 i + R_4 \frac{R_4}{R_4 + R_5} e_1 = 90V$$

The solution can be determined by solving the balance equation following from Kirchhoff's law $u_{AB} - R_{AB}i_N = f(i_N)$. Drawing the load line, determined by the linear part of circuit $u_N = u_{AB} - R_{AB}i_N = 90 - 15,4i_N$ in common plot with the nonlinear characteristic $u_N = f(i_N)$, we get three equilibrium points: P₁, P₂ and P₃ which form three possible solutions of the nonlinear circuit of Fig. 7.5a can be found.

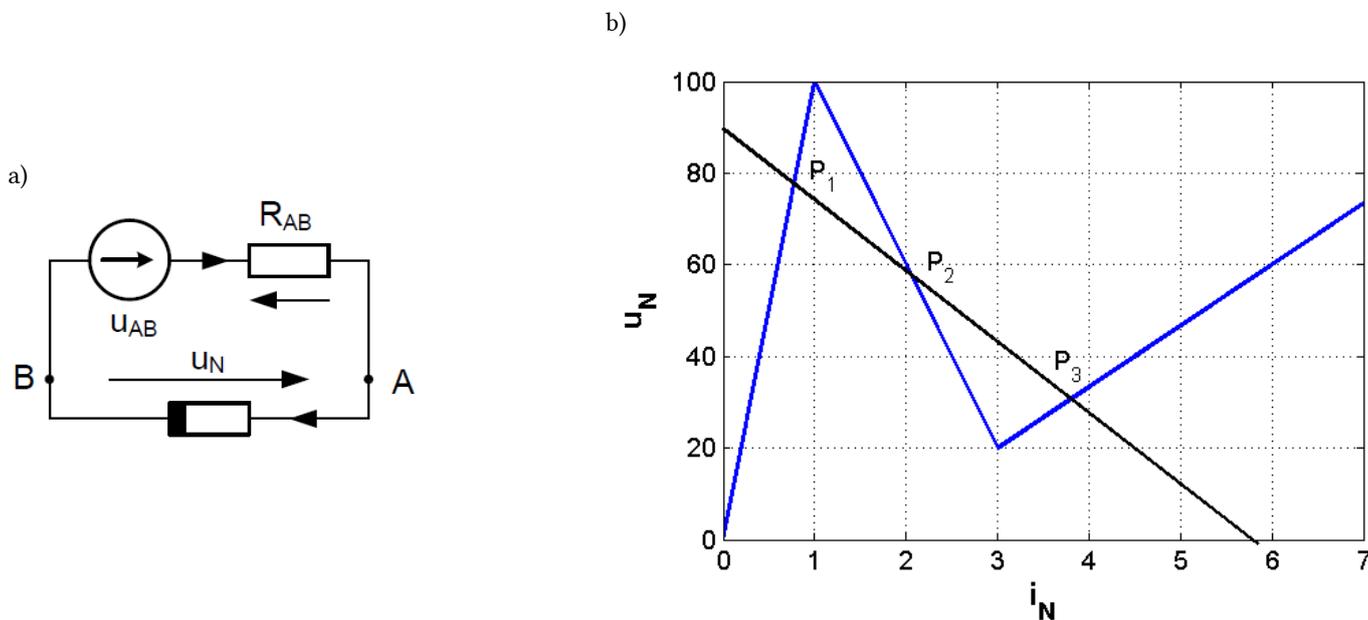


Fig. 7.6. The illustration of PWL solution of the nonlinear circuit: a) the simplified one-loop circuit following from Thevenin theorem, b) the equilibrium points resulting from crossing the load line and nonlinear characteristic.

All three solutions should be calculated subsequently. The linear equation representing the first section with point P₁ is given in the form $u_N=100i_N$. Hence the balance equation is as follows

$$90 - 15.4i_N=100 i_N$$

Solving it we obtain immediately $i_N=0.78A$, $u_N=78V$.

The point P₂ corresponds to the linear equations of the next section of PWL, which can be written in the form $u_N=-40i_N+140$. Hence the balance equation is as follows

$$90 - 15.4i_N=-40i_N+140$$

Solving it we get $i_N=2.03A$, $u_N=58.8V$.

Point P3 corresponds to the linear equations of the next section which is described by the linear equation in the form $u_N=40/3i_N-20$. Hence the balance equations is as follows

$$90 - 15.4i_N=40/3i_N-20$$

Solving it we get the current and voltage of nonlinear element: $i_N=3.83A$, $u_N=31.1V$. As we can see there are three possible solutions of the circuit. The particular value depends on the starting point on the nonlinear element characteristic applied by the algorithm.

7.2.3. Application of PWL in analysis of complex nonlinear circuits

In the case, when few nonlinear elements are included in the circuit, the direct application of Thevenin theorem is impossible. However, PWL representation of nonlinear characteristic is still possible in solution to the problem. Each nonlinear element is represented by the set of PWL representations and the k th PWL section is represented by the connection of linear model of resistor $R(k)$ and ideal voltage $u(k)$ or current $i(k)$ source. It might be the series connection of elements

$$u= R(k) i+ u(k) \tag{7.4}$$

or parallel connection, for which $G(k)=1/R(k)$

$$i= G(k)u+ i(k) \tag{7.5}$$

The values of resistance and source are different for each PWL section. Further procedure of solution takes into account all possible combinations of nonlinear element representation. The number of these combination might be very high, however, in each case we have to solve the linear system of equations, delivering the numerical values of current and voltage (u_N , i_N) of nonlinear elements considered in the combination. If this solution is within the considered region of PWL approximation we accept it as the real solution of the nonlinear circuit, otherwise reject it. To reduce the number of possible combinations we can apply some pre-analysis of the circuit, excluding in advance some combinations.

Example 7.4

The presented above analysis method will be illustrated on the example of the circuit shown in Fig. 7.7. This is the popular circuit implementing the absolute value of the input voltage u . Let us assume the ideal operational amplifier (infinite gain, infinite input impedance and zero output impedance) and ideal diode (short circuit at non-negative voltage between anode and cathode and open circuit for negative value of this voltage).

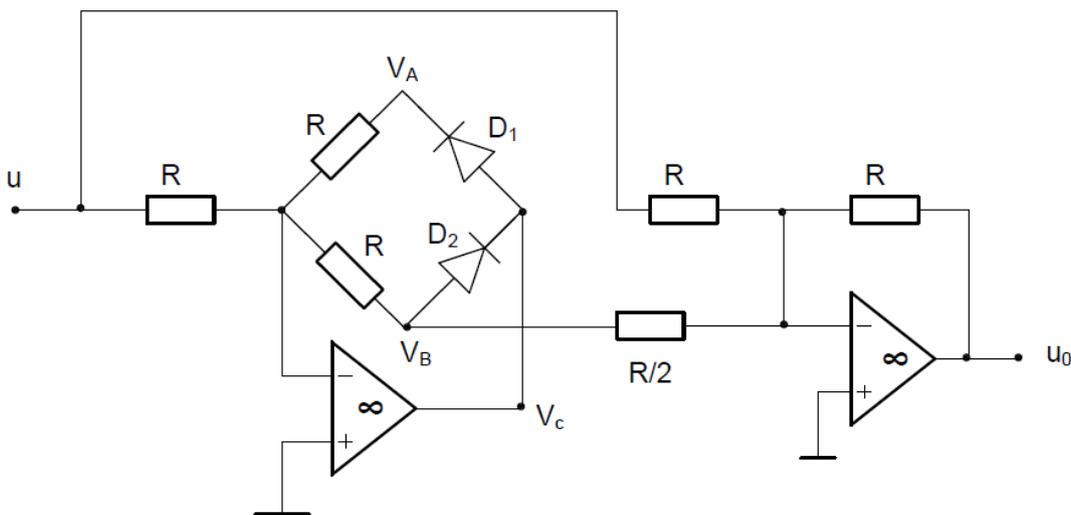


Fig. 7.7. The circuit structure implementing the absolute value of the input signal

Observe that irrespective of polarity of the input voltage u the feedback resistance R of the first amplifier is the

same (at positive polarization of u the diode D_2 is conducting and at negative polarization the conducting diode is D_1). It means, that in both cases V_C is determined by

$$V_C = -\frac{R}{R}u = -u \tag{7.6}$$

Now, two cases might happen in the operation of this circuit.

1. Input voltage $u \geq 0$

In this case we have $V_C \leq 0$, diode D_2 is conducting and represents the short circuit, hence $V_C = V_B = -u$. The output voltage is now defined by the relation

$$u_0 = -2V_B - u = 2u - u = u \tag{7.7}$$

1. Input voltage $u \leq 0$

In this case $V_C \geq 0$, diode D_1 is conducting and represents the short circuit, and diode D_2 is open, hence $V_B = 0$. The output voltage is described now by the relation

$$u_0 = -2V_B - u = 0 - u = -u \tag{7.8}$$

Hence, irrespective of the polarity of the input voltage we have the output signal described as the absolute value of the input signal, i.e.,

$$u_0 = |u| \tag{7.9}$$

Example 7.5

As the second example consider the nonlinear circuit presented in Fig. 7.8. The voltage source E represents the polarization, of the value adjusted by the user. The searched characteristic of the circuit is the input current i as the function of input voltage u . Once again we assume the operational amplifiers and diodes as ideal elements.

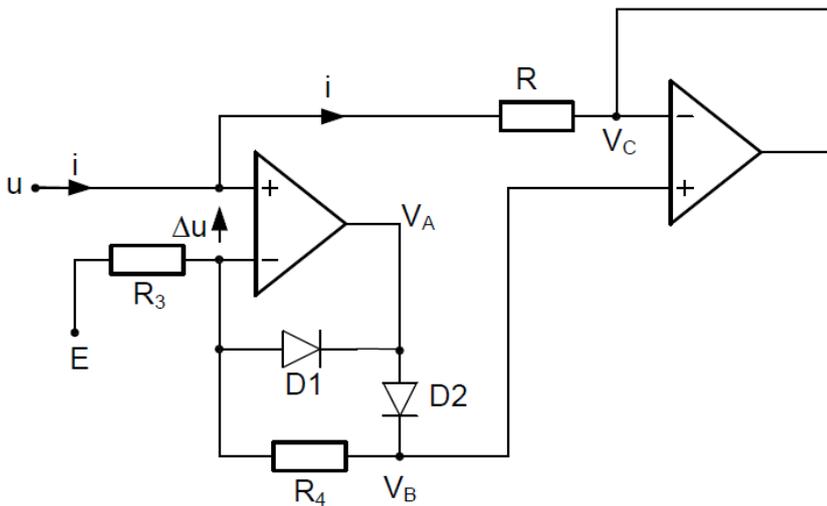


Fig. 7.8. The nonlinear circuit under analysis in example 7.5

In contrast to the previous example the feedback resistance of the first amplifier is either zero (when diode D_1 is conducting) or R_4 (when D_1 is open and D_2 conducting). Consider two cases of the polarity of input voltage u .

1. Input voltage $u \leq E$

In this case the $E > u$, hence $\Delta u = u - E \leq 0$ and $V_A \leq 0$. At such polarization the diode D_1 is conducting and represents the short circuit, the diode D_2 is open. At ideal amplifier the potential of both input terminals of operational amplifier is the same and equal u . This voltage is also transferred to the output V_A of amplifier. Resistor R_4 does not conduct current, hence $V_B = u$. This voltage is transferred to the point C of the second amplifier resulting in $V_C = V_B = u$. Hence the input current of the circuit is equal

$$i = \frac{1}{R}(u - V_C) = 0 \quad (7.10)$$

 1. Input voltage $u \geq E$

In this case we have $\Delta u = u - E \geq 0$ and $V_A \geq 0$. At such polarization diode D_2 is conducting and represents the short circuit ($V_A = V_B$), and diode D_1 is open. The resistor R_4 represents now the feedback of the first amplifier. As a result we get the simplified form of first amplifier connection presented in Fig. 7.9a. The Mason graph corresponding to this circuit is shown in Fig. 7.9b.

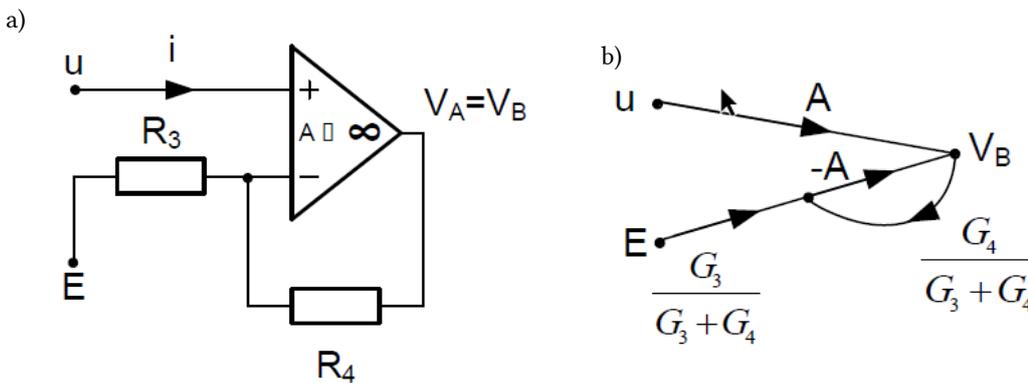


Fig. 7.9. The simplified circuit for the second polarization $u \geq E$ (a) and SFG representation of this circuit (b).

Assuming the gain A tending to infinity, from the Mason gain formula, we get

$$V_A = V_B = u \left(1 + \frac{G_3}{G_4} \right) - \frac{G_3}{G_4} E \quad (7.11)$$

Taking into account $V_C = V_B$ the input current of the circuit can be described by

$$i = \frac{1}{R}(u - V_C) = -\frac{G_3}{RG_4}(u - E) = -\frac{R_4}{RR_3}(u - E) \quad (7.12)$$

Hence, the final form of the input current can be presented in the form

$$i = \begin{cases} 0 & \text{dla } u \leq E \\ -\frac{R_4}{RR_3}(u - E) & \text{dla } u \geq E \end{cases} \quad (7.13)$$

Fig. 7.10 presents the current-voltage characteristic of the considered circuit. The slope of the characteristic in the range $u \geq E$ is equal $G = -R_4/(RR_3)$. It represents the negative conductance.

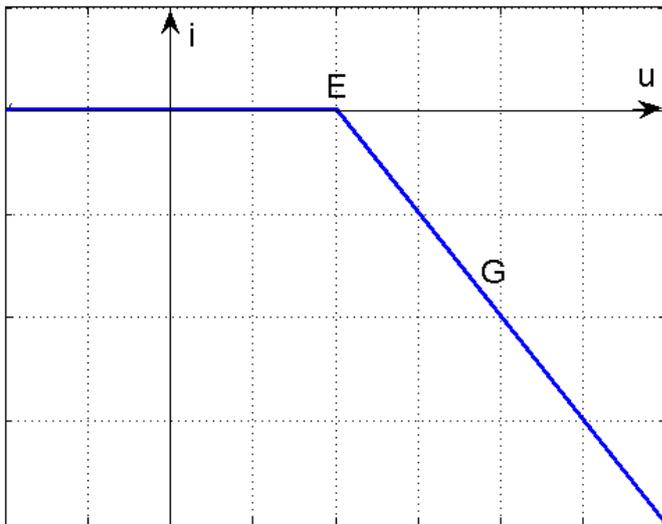


Fig. 7.10. The current-voltage characteristic of the circuit of Fig. 7.8

7.2.4. Numerical methods of solution of nonlinear equations

The nonlinear circuit is represented in the general form

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{7.14}$$

where vector \mathbf{x} represents the unknown currents and voltages of the branches. The size of this vector depends on the number of nonlinear elements. Analytic methods via algebraic approach do not exist in general case. Therefore, the numerical methods can be used instead to approximate the solution of equations. They construct successive approximations in an iterative way, that converge to the exact solution of an equation or system of equations.

The simplest approach to the problem is the so called simple iteration method. According to it the eq. (7.14) is transformed to [3,6,7]

$$\mathbf{x} = \mathbf{x} + \mathbf{f}(\mathbf{x}) \tag{7.15}$$

The succeeding iterations are defined in the form

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{f}(\mathbf{x}_n) \tag{7.16}$$

and lead finally to the solution in the point \mathbf{x}_{n+1} , in which $\|\mathbf{x}_{n+1} - \mathbf{x}_n\| \leq \varepsilon$, where ε is the assumed tolerance. This method is usually very slow and shows linear convergence, in which the error is reduced by a constant factor every iteration, for example, the number of accurate digits increases by 1 every iteration. Unfortunately, its convergence strictly depends on the initial point in calculation. In start far from the solution the method may even diverge.

Much better are the methods of quadratic convergence, in which the number of accurate digits doubles every iteration. Such property is associated with Newton method [3,7,16]. Newton method is the most powerful approach that is used to solve the equation $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. This method originates from the Taylor's series expansion of the function $\mathbf{f}(\mathbf{x})$ about the actual point \mathbf{x}_j . Limiting the expansion to the first order term we have

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_j) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_j) + \dots \tag{7.17}$$

Assuming $\mathbf{x} = \mathbf{x}_{j+1}$ as the approximate solution to the system of equations we may write

$$\mathbf{0} = \mathbf{f}(\mathbf{x}_j) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_{j+1} - \mathbf{x}_j) \quad (7.18)$$

Rearranging the (7.18) we obtain the next approximation of the solution, giving us:

$$\mathbf{x}_{j+1} = \mathbf{x}_j - \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]^{-1} \mathbf{f}(\mathbf{x}_j) \quad (7.19)$$

The function $\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ is the Jacobian matrix calculated in the actual point of solution $\mathbf{x}=\mathbf{x}_j$. This matrix is defined as follows

$$\mathbf{J} = \begin{bmatrix} \partial f_1 / \partial x_1 & \partial f_1 / \partial x_2 & \cdots & \partial f_1 / \partial x_n \\ \partial f_2 / \partial x_1 & \partial f_2 / \partial x_2 & \cdots & \partial f_2 / \partial x_n \\ \vdots & \vdots & \vdots & \vdots \\ \partial f_n / \partial x_1 & \partial f_n / \partial x_2 & \cdots & \partial f_n / \partial x_n \end{bmatrix} \quad (7.20)$$

Newton's method converges quadratically. When carrying out this method the system converges quite rapidly once the approximation is close to the actual solution of the nonlinear system. Hence, good initial guess is extremely important in nonlinear solvers.

In general Newton's method may require less iterations, compared to another methods with a lower rate of convergence, to reach the solution. However, when the system does not converge, this is an indicator that an error in the computations has occurred, or a solution may not exist.

One of the advantages of Newton's method is that it is not too complicated in form and can be used to solve a variety of problems. The major disadvantage associated with Newton's method, is that the Jacobian matrix $\mathbf{J}(\mathbf{x})$, as well as its inversion, has to be calculated for each iteration. Calculating both the Jacobian matrix and its inverse can be quite time consuming depending on the size of the system is. Another problem, that we may be challenged with when using Newton's method, is that it may fail to converge. If Newton's method fails to converge this will result in an oscillation between points.

Relatively simple is the graphical interpretation of the Newton's method. It will be shown for the scalar equation $f(x)=0$. For such simplified problem the solution can be written as follows

$$x_{j+1} = x_j - \left[\frac{\partial f}{\partial x} \right]^{-1} f(x_j) \quad (7.21)$$

Its graphical interpretation is depicted in Fig. 7.11. The new solution point x_{j+1} is placed on the crossing point of the slope of the curve $f(x_j)$ with the horizontal coordinate x . Repeating this procedure in the succeeding iterations we will end in the true solution indicated in the figure as point Q.

It should be noted that Newton's method does not provide automatically the convergence to the true solution. If the starting point is far from the true solution the process may be divergent or the process may oscillate around the equilibrium point. In the latter case we can stabilize the process by introducing the so called damping coefficient λ of varying value. In such case the algorithm is modified as follows [4]

$$\mathbf{x}_{j+1} = \mathbf{x}_j - \lambda \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]^{-1} \mathbf{f}(\mathbf{x}_j) \quad (7.22)$$

at the value of λ in the range $[0 - 1]$. However, the most suggested solution is to find starting point as close to the expected solution as possible. Quite often, the graphical methods of analysis are employed, to find the starting point for further iterations.

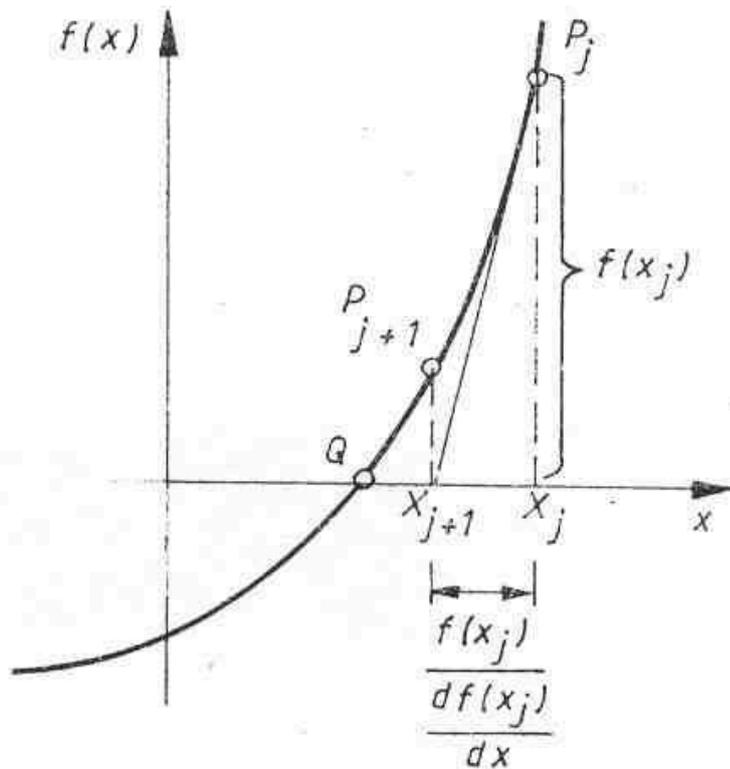


Fig. 7.11 Interpretation of the Newton's method for scalar equation $f(x)=0$.

7.3 Application of Matlab in solving the system of nonlinear equations

Matlab has many functions (grouped in the optimization toolbox [11]) which can be employed for solution of the system of nonlinear equations, defined in the standard form $\mathbf{f}(\mathbf{x})=0$. In this section we will show the application of the function *fsolve* in solution of the set of nonlinear function written in the vector form

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix} = \mathbf{0} \tag{7.23}$$

The unknown variables x_n are grouped in the n -dimensional vector \mathbf{x} . Function *fsolve* can be used with the additional parameters p_1, p_2, \dots defined by the user. Full form of calling this function is as follows

$$[x, fval, exitflag, output, jacobian] = fsolve(fun, x0, options, p1, p2, \dots)$$

and short form requires only to have the file *fun*, defining the set of nonlinear equations $\mathbf{f}(\mathbf{x})$ and starting point x_0 . The function accepts also the explicitly defined Jacobian matrix $\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$.

Example 7.5

As an example consider the application of the function *fsolve* [11] to analyze the nonlinear resistive circuit of the structure presented in Fig. 7.12. The nonlinear elements and values of the sources are as follows

$$i_1 = 3u_1^3, \quad i_2 = 5u_2^5 + 2u_2, \quad u_3 = i_3^3 - 9i_3, \quad e_2 = 1V, \quad e_3 = 5V$$

Find the solution for the voltages u_1 , u_2 and the current i_3 of nonlinear elements for the excitation of e_1 changing from -2V to 3V.

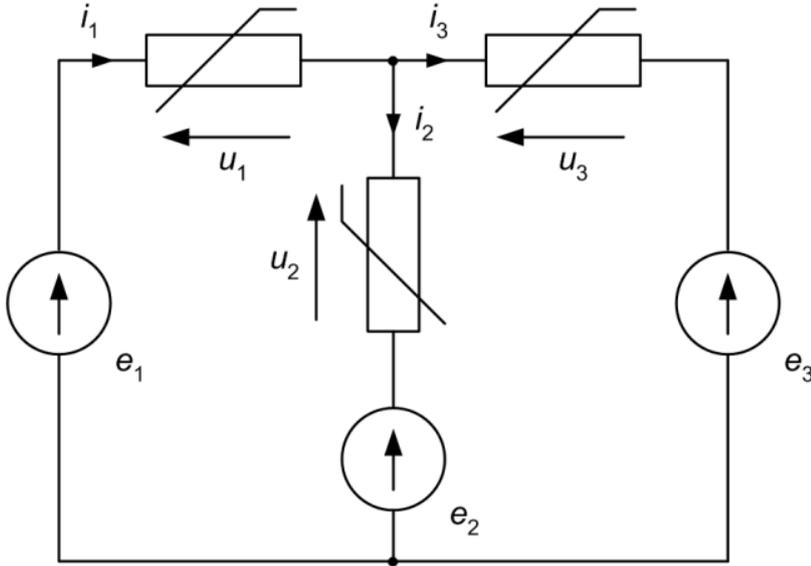


Fig. 7.12. The nonlinear resistive circuit structure

First, we have to find the description of the circuit in the standard form $\mathbf{f}(\mathbf{x})=0$, which is required by the Matlab program. From the Kirchhoff's law we have

$$\begin{aligned} i_1 &= i_2 + i_3 \\ e_1 &= u_1 + u_2 + e_2 \\ e_1 &= u_1 + u_3 + e_3 \end{aligned} \tag{7.24}$$

Rearranging the (7.24) we obtain the required form $f_i(\mathbf{x})=0$ for $i=1, 2, 3$.

$$\begin{aligned} -3u_1^3 + 5u_2^5 + 2u_2 + i_3 &= 0 \\ u_1 + u_2 - e_1 + e_2 &= 0 \\ u_1 + i_3^3 - 9i_3 - e_1 + e_3 &= 0 \end{aligned} \tag{7.25}$$

The vector \mathbf{x} is formed by the voltages of the first and second nonlinear element and the current of the third one, i.e., $\mathbf{x}=[u_1, u_2, i_3]^T$. The main program lines and the function file *objnonl.m* defining the function $\mathbf{f}(\mathbf{x})$ are presented below. The change of the excitation e_1 is done in the program through the additional parameter p_1 controlling the value of e_1 .

```
% Main program lines
options=optimset('LargeScale','off','MaxFunEvals', 1000)
p1=-2:.2:3;
x0=[0 0 0];
np=length(p1);
for i=1:np
e1=p1(i);
```

```

[x,fval]=fsolve('objnonl',x0,options,e1)
y(i,:)=x;
end
plot(p1,y(:,1),'-',p1,y(:,2),'--',p1,y(:,3),'-.')
grid
title('Solution of nonlinear circuit')
legend('u1','u2','i3')
xlabel('e_1')
ylabel('u1, u2, i3')
function f=objnonl(x,e1)
% File defining f(x)=0
u1=x(1); u2=x(2); i3=x(3);
e2=1; e3=5;
f(1)=-3*u1^3+5*u2^5+2*u2+i3;
f(2)=u1+u2-e1+e2;
f(3)=u1+i3^3-9*i3-e1+e3;

```

The obtained results presenting the changes of u_1 , u_2 and the current i_3 as a function of the input voltage e_1 are depicted in a graphical form in Fig. 7.13.

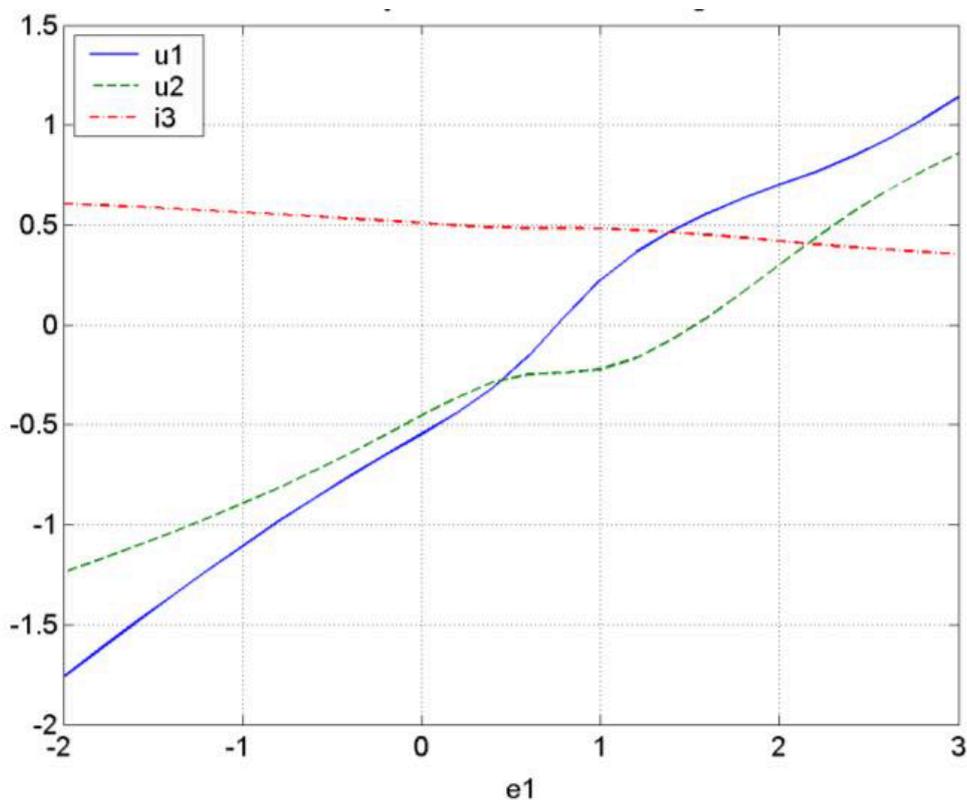


Fig. 7.13. The changes of the voltages and current of nonlinear element versus e_1 .

Example 7.6

As the next example consider the application of *fsolve* to the solution of the magnetic circuit presented in Fig. 7.14a. The concept of a magnetic circuit exploits a one-to-one correspondence between the equations of the magnetic field in the ferromagnetic material to that of an **electrical circuit**. Using this concept the magnetic fields of complex devices (like the circuit in Fig. 7.14a) can be quickly solved using the methods and techniques developed for electrical circuits.

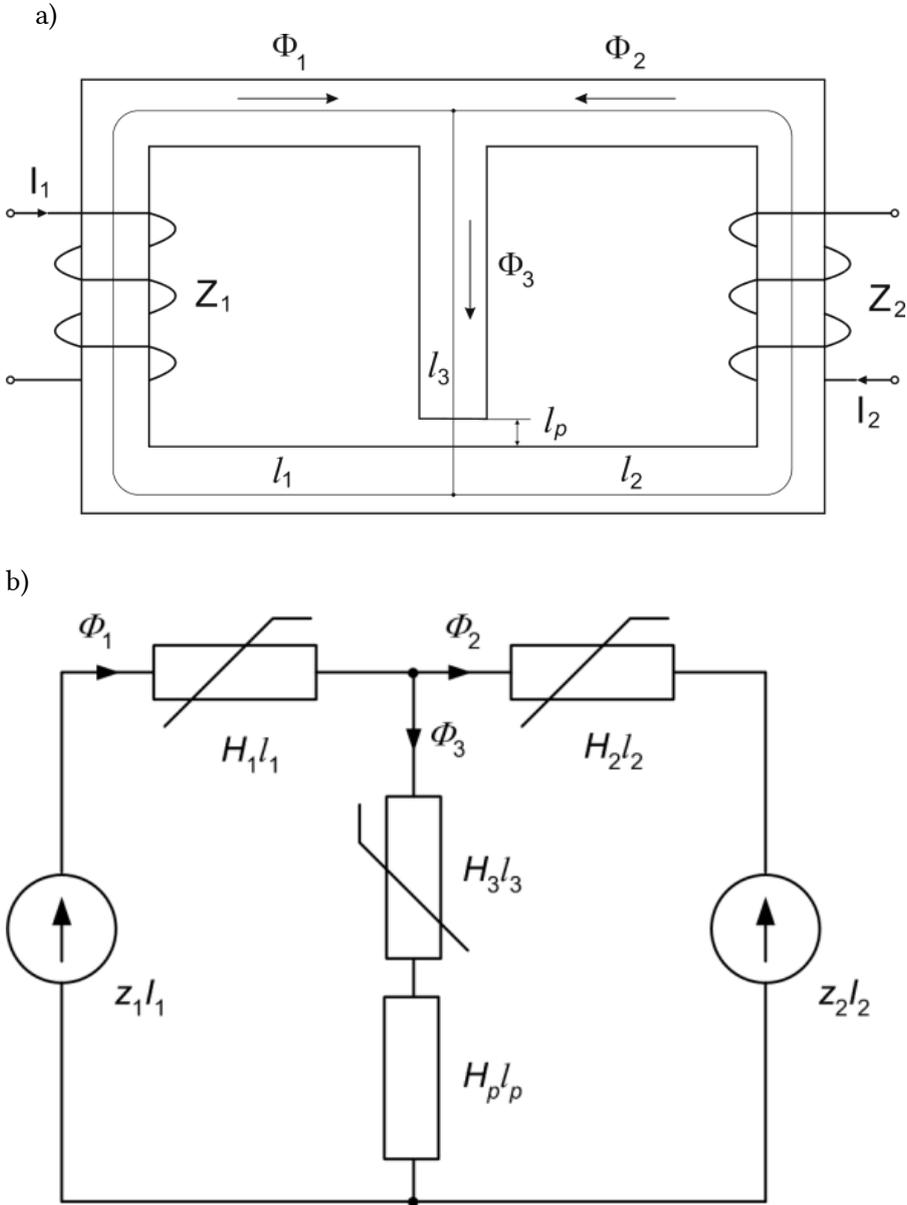


Fig. 7.14. Structure of the magnetic circuit excited by the currents I_1 and I_2 (a) and its equivalent nonlinear resistive circuit (b)

The magnetic circuit in the example is made up of the closed loop paths containing three branches of **magnetic cores** consisting of **ferromagnetic materials** (iron) and small air gap in the middle branch. Assume the given magnetization curve and all dimensions (length and cross areas) of the magnetic cores and air gap as well as the values of currents I_1 and I_2 and number of wires z_i of the coils.

In the first step we have to describe the magnetic circuit by proper set of equations. The magnetic circuit can be treated as a special type of nonlinear electrical circuit in which the variables are: magnetic fluxes Φ_i , magnetic flux density B_i , magnetic field H_i , the magnetic potential difference $H_i l_i$ of the branches and

magnetomotive force (MMF) defined as an excitation $zili$ (equivalent to the electrical voltage source). Similar to the way that electromotive force (EMF) drives a current in electrical circuits, magnetomotive force (MMF) drives magnetic flux through magnetic circuits.

Specific for magnetic elements is the magnetic curve $B=F(H)$, relating magnetic flux density B to the magnetic field H . In numerical calculations this curve is described in a discrete way in the form of the given points (H_i, B_i) . To find their values between these points we will use the *spline* function of Matlab [10]. Third order spline is used to calculate any value (H_i, B_i) in the calculations. It is called in the way

$$B=\text{spline}(HH, BB, H)$$

where the variables HH, BB represent the given (known) points of magnetization curve. The H is the actual value of magnetic field for which the magnetic flux density B is searched.

In preparation of equations describing the magnetic circuit we will use its equivalent electrical circuit (Fig. 7.14b), in which the role of current fulfills flux Φ_i , the branch voltage – the magnetic potential difference $Hili$ of the element of the length li in which the magnetic field Hil is present, the voltage source – the magnetomotive force $\Theta_i = z_i I_i$ (z_i – number of turns of the coil). Applying Kirchhof's law for such defined electrical equivalent we may write

$$\begin{aligned} H_1 l_1 + H_3 l_3 + H_p l_p - z_1 I_1 &= 0 \\ H_2 l_2 + H_3 l_3 + H_p l_p - z_2 I_2 &= 0 \\ \Phi_1 + \Phi_2 - \Phi_3 &= 0 \end{aligned} \quad (7.26)$$

All variables $B_i = F(H_i)$ for $i = 1, 2, 3$ are fully defined by the magnetic curve, common to all ferromagnetic branches, and $H_p = B_3 / \mu_0 = F(H_3) / \mu_0$. The fluxes are equal $\Phi_i = S_i B_i = S_i F(H_i)$ for $i = 1, 2, 3$. As a result we get the following system of nonlinear equations

$$\begin{aligned} H_1 l_1 + H_3 l_3 + l_p F(H_3) / \mu_0 - z_1 I_1 &= 0 \\ H_2 l_2 + H_3 l_3 + l_p F(H_3) / \mu_0 - z_2 I_2 &= 0 \\ S_1 F(H_1) + S_2 F(H_2) - S_3 F(H_3) &= 0 \end{aligned} \quad (7.27)$$

This system of equations represents the general vector form $\mathbf{f}(\mathbf{x})=\mathbf{0}$ which is needed by the function *fsolve*. Vector \mathbf{x} of unknown variables contains three variables $\mathbf{x}=[H_1, H_2, H_3]T$. The Matlab program implementing numerical calculations related to the problem is presented below.

```
% Main program
global HH
global BB
global p
H0=[0 0 0];
p=[ -1 -0.5 0 0.5 1 1.5 2];
% Magnetic curve HH-BB
HH = [-4500 -4000 -3500 -3000 -2500 -2000 -1500 -1000 -500 ...
0 500 1000 1500 2000 2500 3000 3500 4000 4500 5000 5500];
BB = [-1.63 -1.60 -1.56 -1.51 -1.45 -1.38 -1.25 -1.0 -0.62 ...
0 0.62 1.0 1.25 1.38 1.45 1.51 1.56 1.60 1.63 1.65 1.66];
% Loop for changing the excitation p
for i=1:length(p)
```

```

i1=p(i)
H=fsolve(@fun_magn, H0,optimset('fsolve'),i1);
y1(i,:)=H;
y2(i,:)=spline(HH,BB,H);
end
% Graphical results
subplot(211),plot(p,y1(:,1),'-', p,y1(:,2),'--', p,y1(:,3),'-.')
grid, xlabel('i1'), ylabel('H1, H2, H3')
title(' Results of analysis of magnetic circuit')
subplot(212),plot(p,y2(:,1),'-', p,y2(:,2),'--', p,y2(:,3),'-.')
grid, xlabel('i1'), ylabel('B1, B2, B3')
function f=fun_magn(H, i1)
global HH, global BB
% Data of magnetic circuit
l1=0.20; l2=0.10; l3=0.20; lp=0.001;
s1=0.001; s2=0.002; s3=0.001;
z1=1000; z2=500; i2=1.5; mi0=4*pi*1e-7;
% Magnetic flux density
B(1)=spline(HH, BB, H(1));
B(2)=spline(HH, BB, H(2));
B(3)=spline(HH, BB, H(3));
% Equations of the circuit f(x)=0
f(1) = H(1)*l1 + H(3)*l3 + lp*B(3)/mi0 - z1*i1;
f(2) = H(2)*l2 + H(3)*l3 + lp*B(3)/mi0 - z2*i2;
f(3) = s1*B(1) + s2*B(2) - s3*B(3);

```

The results of circuit analysis in the form of dependence of magnetic field H_i and flux density B_i ($i=1, 2, 3$) for different values of excitation i_1 are presented in Fig. 7.15. The least nonlinearity is observed in the third branch (H_3, B_3), containing the air gap.

7.4 Exercises

1. Find all solutions of the nonlinear resistive circuit given below

Assume characteristic $u_N=f(i_N)$ of the nonlinear element as shown in Fig. 7.16b and $E=100\text{V}$, $R_1=10\Omega$, $R_2=40\Omega$.

Solution:

Let us apply Thevenin method of solution. The equivalent Thevenin circuit structure is presented in Fig. 7.16c.

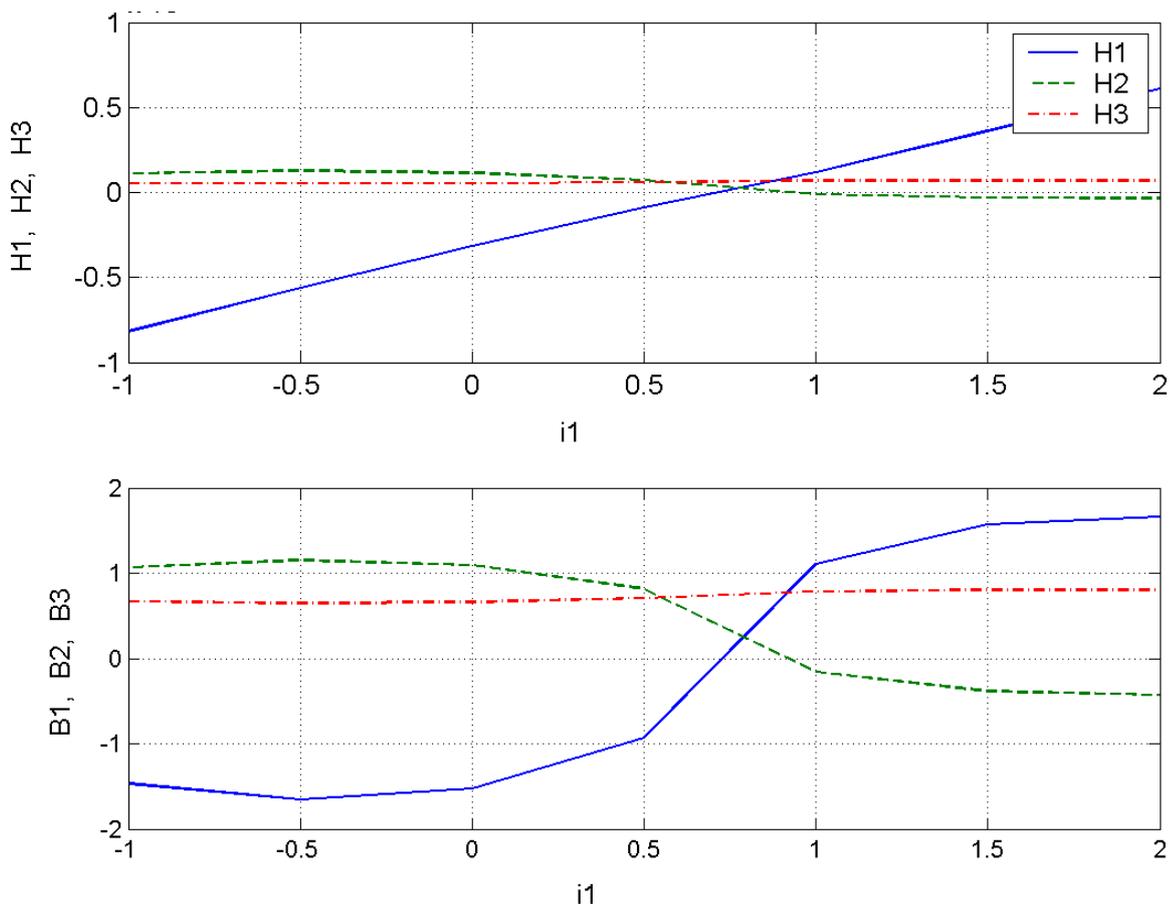


Fig. 7.15. The results of analysis of magnetic circuit as a function of excitation i_1 : a) magnetic field in three branches of magnetic core (scale $\times 10^4$), b) magnetic flux density in these branches.

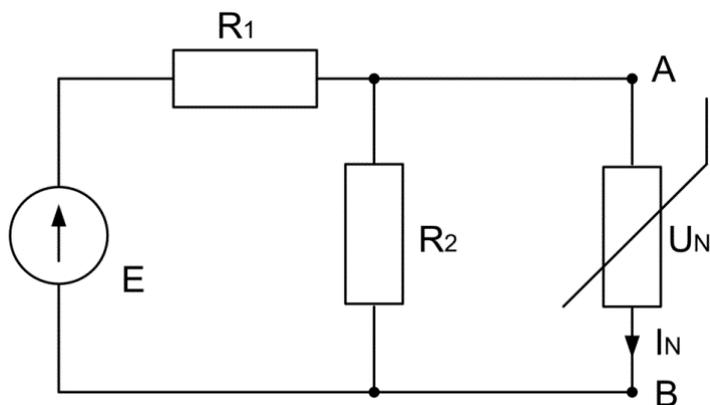


Fig. 7.16a The nonlinear circuit structure

The values of equivalent elements are as follows:

$$U_{AB} = E \frac{R_2}{R_1 + R_2} = 80V$$

$$R_{AB} = \frac{10 \cdot 40}{50} = 8\Omega$$

The operating points of the circuit require satisfying the following equation

$$U_{AB} - R_{AB}i_N = U_N(i_N)$$

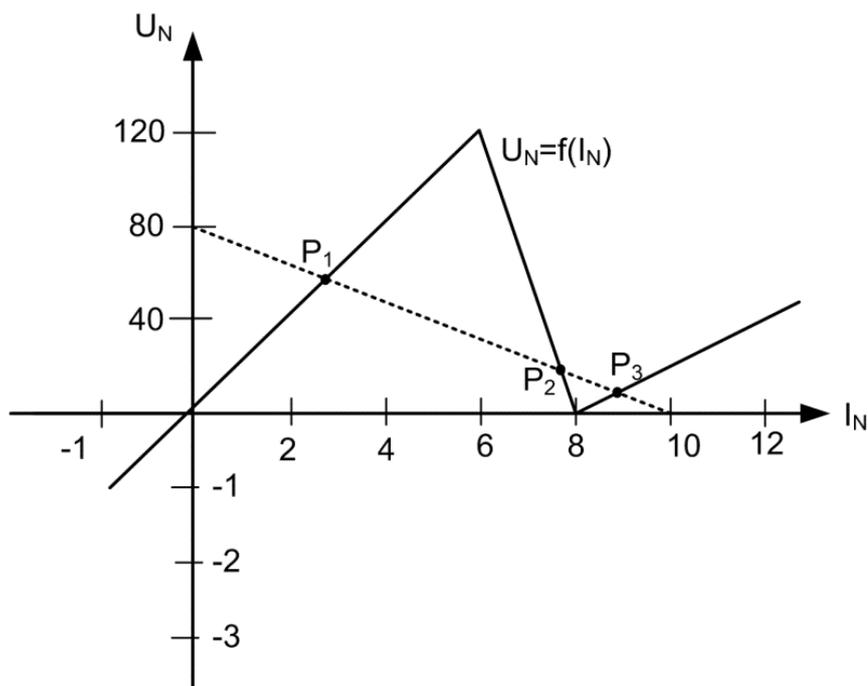


Fig. 7.16b The nonlinear element characteristic $u_N=f(i_N)$ and load line (dotted)

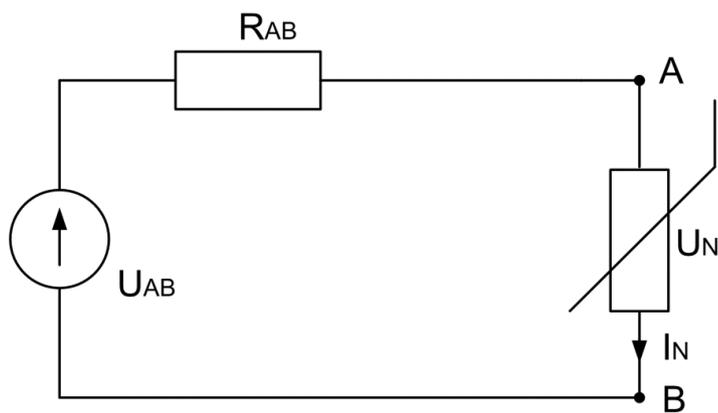


Fig. 7.16c Thevenin equivalent of the circuit of Fig. 7.1a.

The left side of the equation represent the load line and the right side the characteristic of nonlinear element. The load line is described by the equation

$$U_N = 80 - 8I_N$$

Their crossing will result in three solution points: P_1 , P_2 and P_3 . These points are located on three different linear sections of the characteristic. Hence we get three solutions following from the appropriate equations.

1. A) Solution P_1

$$U_N = \frac{120}{6}I_N = 80 - 8I_N \rightarrow I_N = \frac{80}{28} = 2.86A \rightarrow U_N = 57.14V$$

1. B) Solution P_2

$$U_N = \frac{-120}{2} I_N + 480 = 80 - 8I_N \rightarrow I_N = \frac{400}{52} = 7.69A \rightarrow U_N = 18.46V$$

1. C) Solution P3

$$U_N = \frac{40}{4} I_N - 80 = 80 - 8I_N \rightarrow I_N = \frac{160}{18} = 8.89A \rightarrow U_N = 8.89V$$

1. Describe the nonlinear circuit given below by the set of nonlinear equations. Assume

$$U_1 = a_1 I_1 + a_3 I_1^3$$

$$U_2 = a_5 I_2^5$$

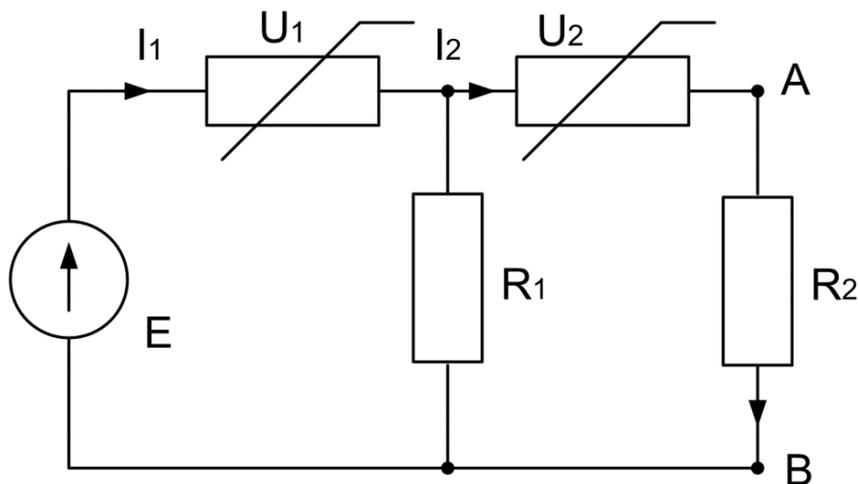


Fig. 7.17 The nonlinear circuit structure

Solution:

From Kirhhoff's equations we get

$$E = a_1 I_1 + a_3 I_1^3 + R_1 (I_1 - I_2)$$

$$0 = R_1 (I_1 - I_2) - a_5 I_2^5 - R_2 I_2$$

After rearranging them we get their final form $f(\mathbf{x})=0$

$$(R_1 + a_1) I_1 + a_3 I_1^3 - R_1 I_2 - E = 0$$

$$R_1 I_1 - (R_1 + R_2) I_2 - a_5 I_2^5 = 0$$

1. Calculate mean value of the current in the circuit shown below. Assume $e(t)=100\sin(\omega t)$, $R=100\Omega$ and diode as the nonlinear resistor of the resistance in positive polarity equal 0.5Ω and for negative polarity $10k\Omega$.

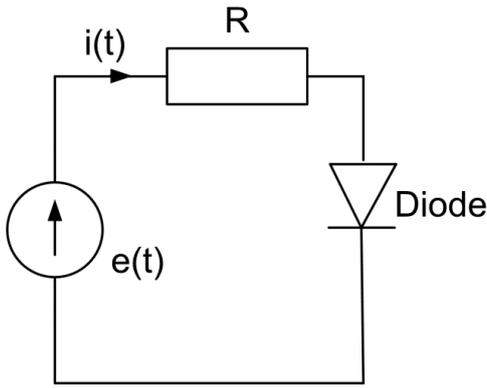


Fig. 7.18 The circuit structure with diode

Solution:

The current in conducting state of diode (positive polarity for $0 \leq \omega t \leq \pi$) is equal

$$i(t) = \frac{100}{100 + 0.5} \sin(\omega t) = 0.995 \sin(\omega t)$$

Similarly the current at negative polarity (for $\pi \leq \omega t \leq 2\pi$) is equal

$$i(t) = \frac{100}{100 + 10000} \sin(\omega t) = 0.01 \sin(\omega t)$$

The mean value I of the current over the period of sinusoid is equal

$$I = \frac{1}{2\pi} \left(\int_0^{\pi} 0.995 \sin(\omega t) d\omega t + \int_{\pi}^{2\pi} 0.01 \sin(\omega t) d\omega t \right) = \frac{0.995}{\pi} - \frac{0.01}{\pi} = 0.313$$

1. Nonlinear circuit is described by the set of equations

$$2x_1 + x_1 x_2 + 3x_2^2 - 1 = 0$$

$$-x_1 + 2x_1 x_2^2 + 0.5x_2^3 + 2 = 0$$

Apply one step of Newton solution method at starting point: $x_1(0)=1, x_2(0)=0$.

Solution:

The first point is to calculate the elements of the Jacobi matrix \mathbf{J} at the starting point. This matrix is in our case described by

$$\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

Performing these calculations at starting point we get

$$\mathbf{J} = \begin{bmatrix} 2 + x_2 & x_1 + 6x_2 \\ -1 + 2x_2^2 & 4x_1 x_2 + 3/2x_2^2 \end{bmatrix}_{|x_1=1, x_2=0} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\mathbf{J}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{f}(\mathbf{x}(0)) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence the next step solution of the nonlinear system equations is of the form

$$\mathbf{x}(1) = \mathbf{x}(0) - \mathbf{J}^{-1}\mathbf{f}(\mathbf{x}(0)) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

7.5 Basic definitions

Field strength – the measure of intensity of a magnetic field H expressed in amperes per meter (A/m). It is related to magnetic flux density B by the relation $B=\mu H$.

Flux density – the measure B of magnetic field expressed in Newton-meters per ampere (Nm/A), also called tesla (T).

Graphical method of analysis – the method of analysis of nonlinear circuit. It operates directly on the characteristics of the elements. The analysis aims on simplification of the circuit by finding equivalent characteristics of series and parallel connected elements.

Jacobian matrix – matrix named after Jacoby is the **matrix** of all first-order **partial derivatives** of a **vector-valued function**. In the case of **square matrix**, both the matrix and its **determinant** are commonly referred to as the Jacobian.

Magnet – a piece of iron, or of certain other materials, that attracts or repels other pieces of iron.

Magnetic field – the **vector field** that describes the magnetic influence of electrical currents and magnetized materials. It includes the area in which the pull of a magnet, or thing acting like a magnet, is felt.

Magnetic flux – the magnetic flux (denoted by Φ) through a surface is the surface **integral** of the normal component of the **magnetic field** B passing through that surface. The **SI unit** of magnetic flux is the **weber** (Wb), equivalent to volt-seconds (Vs).

Magnetic potential difference – the product of magnetic field strength H and the length of the branch l in magnetic circuit.

Magnetomotive force (MMF) – the excitation zI (source) in magnetic circuits (equivalent to the electrical voltage source).

Newton-Raphson method – the method named after **Isaac Newton** and **Joseph Raphson**, is a method for finding successively better approximations to the **roots** (or zeroes) of a **real-valued function**.

Nonlinear circuit – the electrical circuit containing at least one nonlinear element. In contrast to linear circuits the superposition rule is not valid in such circuit. Moreover, nonlinear circuit may possess many solutions, not a single one as it is in linear circuits.

PWL approximation – an approximation to a known curve by sampling the curve and interpolating linearly between the points.

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